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Extensions of the Laplace cascade method

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EXTENSIONS OF THE LAPLACE CASCADE METHOD

JOHN HILARY BILLINGS

EXTENSIONS OF THE LAPLACE CASCADE METHOD

by

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A P R O V A L   S H E E T

Title of Thesis: Extensions of the Laplace Cascade Method

Name of Candidate: John Hilary Billings
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ABSTRACT

Title of Thesis: Extensions of the Laplace Cascade Method.

John Hilary Billings, Doctor of Philosophy, 1960.

Thesis directed by: Research Professor Joaquin B. Diaz.

The Laplace cascade method is concerned with second order linear hyperbolic equations of the form

$$(1)\quad u_{xy} + a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)\, u = 0.$$

The substitutions $u_1 = u_y + au$ and $u_{-1} = u_x + bu$ lead to the equations

$$(2)\qquad u_{1_x} + bu_1 - hu = 0$$

and

$$(3)\qquad u_{1_y} + au_1 - ku = 0,$$

where $h = a_x + ab - c$ and $k = b_y + ab - c$ are the two Darboux invariants. If either h or k vanishes, (1) has been reduced to a system of two first order equations, while if neither vanishes the equation may be cascaded in two directions. Solving (2) (or (3)) for u in terms of u_1(or u_{-1}), and substituting the resulting expression into (1) yields an equation for $u_1(u_{-1})$ of the same form as (1) but with new coefficients, in general. This process may be iterated, forming a chain of equations, until either the original equation reappears, or one of the corresponding invariants vanishes.

An extension of Volterra's product integral to the non-homogeneous system

$$(4)\qquad \frac{du_i(x)}{dx} = \sum_{j=1}^{n} \Big(a_{ji}(x)u_j(x)\Big) + f_i(x),$$

$$u_i(b) = u_{i0} \quad ; \quad i = 1,2,\ldots,n$$

is made first. Then the Laplace method is extended to systems of second order hyperbolic equations, of the form

$$(5)\ \frac{\partial^2 u_i}{\partial x \partial y} + \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} c_{ij} u_j = 0\ ,$$

$$i = 1,2,\ldots,n\ .$$

Matrix notation is promptly introduced, and equation (5) is rewritten as

$$(6)\qquad U_{xy} + AU_x + BU_y + CU = 0\ .$$

The two related matrix invariants for (6) are

$$H = A_x + BA - C$$

$$K = B_y + AB - C\ ,$$

and the chain of equations is developed as for the single equation. If either H or K is identically zero, the resulting two systems of first order equations, may be solved by employing the above-mentioned extension of Volterra's product integral.

The invariance of H and K are discussed in a manner analogous to the invariance of the functions h and k of the single equation. This is followed by a consideration of periodic systems, i.e. systems such

that after j iterations the original equation reappears. This discussion results in two theorems, the first of which is

Theorem I - A system of equations of the form (6) having constant matrix coefficients A and B, can be reduced to the form

$$U'_{xy} = H'U'$$

by a change of variables $U = \Lambda U'$, if an only if $AB = BA$. The second theorem, arising from systems of period two leads to a discussion of the form of the solution to the matrix analog of Liouville's equation

$$\frac{\partial^2 \Theta}{\partial x \partial y} = 2\, e^{\Theta} - 2\, e^{\Theta} .$$

Discussion of the form of the solution when the chain terminates after a finite number of iterations leads to two further theorems which are completely analogous to theorems proved by Darboux for the single equation.

The second extension of the Laplace cascade method is made to the third order linear hyperbolic equation in three independent variables of the form

$$(7) \qquad u_{xyz} + au_{yz} + bu_{xz} + cu_{xy} + du_x + eu_y + fu_z + gu = 0 .$$

Here the number of invariant functions to be consider jumps from two to eighteen. The nature of the "invariance" of these functions is different from that of the second order invariants, in that some of the functions are true invariants, while others can only be considered as quasi-invariants. Four methods of cascading the equations are discussed, each of which requires severe restrictions on the coefficients a, b,...,g. The class of equations,

for which each method will result in a termination of the chain after a finite number of iterations, is explicitly pointed out.

The final extension is a generalization of this third order extension to the n^{th} order linear hyperbolic equation with n independent variables, of the form

$$(8) \quad u_{x_1 x_2 \cdots x_n} + \sum_{i=1}^{n} a_i u_{x_1 \cdots x_{i-1} x_{i+1} \cdots x_n} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij} u_{x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n} + \cdots + b_i u_{x_i} + cu = 0$$

Introducing the linear operator D, so that (8) becomes

$$D(u) = 0,$$

the number of identities which may lead to a decomposition of (8) into a system of two equations of lesser order is computed. The exact number of true invariants is determined to be $n(n-1)$, while upper and lower bounds on the number of quasi-invariants are formulated. Theorem V proves that an invariant h is a true invariant, i.e. invariant under the change of coordinates $u = \lambda(x_1, x_2, \ldots x_n)\, u'$, if

$$h = \frac{\partial a_{(i)}}{\partial x_j} + a_{(i)} a_{(j)} - a_{(ij)}$$

for some $i,j = 1,2,\ldots,n$, $i \neq j$, while any other invariant is, in general not a true invariant, hence a quasi-invariant.

A discussion of earlier extensions is included.

VITA

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SECTION I

INTRODUCTION

The cascade method was devised by Pierre Simon Laplace[1], and investigated in complete detail by Gaston Darboux.[2] Earlier extensions were made by U. Dini,[3] and J. LeRoux[4], and these extensions are discussed briefly in SECTION VI of this paper. The method itself deals with the linear hyperbolic equation with variable coefficients which has the form

$$(1) \qquad \mathcal{L}(u) \equiv \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0,$$

where the coefficients a, b, and c are analytic in a certain domain D, or at least as many times continuously differentiable as we feel is necessary, and $u = u(x,y)$ is a real function of real variables.

Laplace's cascade method begins with the introduction of what are now called the two Darboux invariants, h and k, which are defined by the relations

$$(2) \qquad \begin{aligned} h &= \frac{\partial a}{\partial x} + ab - c, \\ k &= \frac{\partial b}{\partial y} + ab - c. \end{aligned}$$

Then (1) may be written in either of the following forms:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + au\right) + b\left(\frac{\partial u}{\partial y} + au\right) - hu = 0 ;$$

(3)

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + bu\right) + a\left(\frac{\partial u}{\partial x} + bu\right) - ku = 0.$$

We may then consider the obvious substitutions

$$u_1 = \frac{\partial u}{\partial y} + au ,$$

(4)

$$u_{-1} = \frac{\partial u}{\partial x} + bu ,$$

and further reduce (3) to

$$\frac{\partial u_1}{\partial x} + bu_1 - hu = 0 ;$$

(5)

$$\frac{\partial u_{-1}}{\partial y} + au_{-1} - ku = 0 .$$

Now if we should be so fortunate that either function h or k is identically zero, we will have succeeded in reducing our original second order equation to a system of two first order equations, either

$$u_1 = \frac{\partial u}{\partial y} + au ,$$

(6a)

$$\frac{\partial u_1}{\partial x} + bu_1 = 0 ;$$

or

(6b)
$$u_{-1} = \frac{\partial u}{\partial x} + bu ,$$
$$\frac{\partial u_{-1}}{\partial y} + au_{-1} = 0 ,$$

either of which system may be solved by quadratures.

If, however, as is more likely the case neither h nor k is identically zero, all is not lost. To be specific, let us consider the system (6a), as the details are quite similar whichever system we choose.[5] Instead of (6a), we have

(6a')
$$u_1 = \frac{\partial u}{\partial y} + au ,$$
$$\frac{\partial u_1}{\partial x} + bu_1 - hu = 0 .$$

We may integrate the first equation of (6a') to obtain

(7)
$$u = e^{-\int a dy} \left[\int e^{\int a dy} u_1 \, dy + X(x) \right] ,$$

where $X(x)$ is an arbitrary function of x. Substitution of this expression for u into the second equation of (6a') yields

$$\frac{\partial u_1}{\partial x} + bu_1 - he^{-\int a dy} \left[\int e^{\int a dy} u_1 \, dy + X(x) \right] = 0 ,$$

or

$$(8) \qquad \frac{e^{\int a\,dy}}{h}\left[\frac{\partial u_1}{\partial x} + bu_1\right] = \int e^{\int a\,dy}\, u_1\, dy + X(x).$$

Taking the partial derivative of both sides with respect to the variable y gives

$$\frac{e^{\int a\,dy}}{h}\left\{a\left[\frac{\partial u_1}{\partial x} + bu_1\right] - \frac{\partial \log h}{\partial y}\left[\frac{\partial u_1}{\partial x} + bu_1\right] + \frac{\partial^2 u_1}{\partial x \partial y} + b\frac{\partial u_1}{\partial y} + \frac{\partial b}{\partial y}u_1\right\} = e^{\int a\,dy}\, u_1 .$$

After some straightforward algebraic manipulations, this becomes

$$(9) \qquad \frac{\partial^2 u_1}{\partial x\, \partial y} + a_1\frac{\partial u_1}{\partial x} + b_1\frac{\partial u_1}{\partial y} + c_1 u_1 = 0 ,$$

where $a_1 = a - \dfrac{\partial \log h}{\partial y}$, $b_1 = b$, and

$$c_1 = c - \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - b\frac{\partial \log h}{\partial y} .$$

We observe that (9) is of the very same form as (1), but with new coeffients a_1, b_1, and c_1. Hence we may repeat our original process, in hopes that either of the new invariants h_1 or k_1 will perhaps be zero. If not, we may proceed to evolve a new equation for the variable u_2, and in fact we may iterate the entire procedure as

often as necessary to produce a chain, or cascade of equations, in the hope that at some point the iteration will stop because one of the invariants will become zero. This will enable us to solve a first order system by quadratures, and by tracing our way back through the chain, we can easily solve the original equation (1).

As indicated previously, we may do an analogous procedure with system (6b), producing a cascade of equations in the "opposite direction". Darboux pointed out that if we followed the h substitution with a k substitution, we would not produce a new chain, but would indeed revert to the original equation (1). In fact, if we denote (1) by E, denote the equations obtained from E by use of the functions $h, h_1, h_2, \ldots$ by $E_1, E_2, E_3 \ldots$, and denote those obtained from E by use of the functions $k, k_1, k_2, \ldots$ by $E_{-1}, E_{-2}, E_{-3}, \ldots$, our chain of equations appears as

$$\ldots, E_{-2}, E_{-1}, E, E_1, E_2, \ldots \quad .$$

If we should take any E_n obtained by use of an h_{n-1} function, and attempt to obtain a new equation using a k_n function, we would in fact produce equation E_{n-1}. The same relationships hold for the E_{-n} equations.

Darboux discusses these and many others points regarding the Laplace cascade method, including the nature of the invariance of the h and k functions, periodicity of the h and k functions, and the form of the most general solution obtainable if the chain terminates after a finite number of iterations in either direction. It is the purpose of this thesis to extend this cascade method to larger classes of equations and to carry out similar investigations regarding these new applications of the method.

In particular we will first show how the cascade method can be applied to systems of n linear second order hyperbolic equations in two independent variables, with n dependent variables, discussing the invariant nature of our substitution functions, the general form of a solution when the chain terminates after a finite number of iterations, and the solution of some typical systems. We will consider the system

$$(10) \qquad \frac{\partial^2 u_i}{\partial x \partial y} + \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} c_{ij} u_j = 0, \quad i = 1,2,\ldots,n \; ,$$

where $a_{ij} = a_{ij}(x,y)$, $b_{ij} = b_{ij}(x,y)$, $c_{ij} = c_{ij}(x,y)$ are continuously differentiable as often as necessary, and the $u_i = u_i(x,y)$ are real functions of real variables. We will first express this equation in matrix form, letting $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$,

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}, \qquad (o) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \qquad 0 = \begin{bmatrix} 0\,0 & \ldots & 0 \\ 0\,0 & \ldots & 0 \\ & \ldots & \\ 0\,0 & \ldots & 0 \end{bmatrix} .$$

Then (10) may be written as

$$(11) \qquad U_{xy} + AU_x + BU_y + CU = (0)$$

where $U_{xy} = \begin{bmatrix} \frac{\partial^2 u_1}{\partial x \partial y} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial^2 u_n}{\partial x \partial y} \end{bmatrix}$, $U_x = \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial u_n}{\partial x} \end{bmatrix}$, $U_y = \begin{bmatrix} \frac{\partial u_1}{\partial y} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial u_n}{\partial y} \end{bmatrix}$,

Introducing the two Darboux $n \times n$ matrix invariants

$$H = A_x + BA - C,$$

$$K = B_y + AB - C,$$

we will be able to proceed with development of chains and solutions by the cascade method.

Our second extension of this method will be to a single third-order linear hyperbolic equation in three independent variables, of the form

$$(12) \qquad u_{xyz} + a(x,y,z)u_{yz} + b(x,y,z)u_{xz} + c(x,y,z)u_{xy} + d(x,y,z)u_x + e(x,y,z)u_y + f(x,y,z)u_z + g(x,y,z)\,u = 0.$$

Since we will be dealing in three independent variables, we will see that three chains will result, in the "x-direction", the "y-direction", and the "z-direction". Also, since the equation is of third order, we will see that eighteen "invariant" functions must be introduced. We will note that the invariance of these functions will not be of the same nature as that of the corresponding functions in the second order, two

must be placed on the coefficients $a(x,y,z), \ldots, g(x,y,x)$ before a chain of equations can be developed.

Finally we will generalize the results indicated in the preceding paragraph to the single linear hyperbolic equation of n^{th} order in n variables, which has the form

$$\frac{\partial^n u}{\partial x_1 \partial x_2 \cdots \partial x_n} + \sum_{i=1}^{n} a_i \frac{\partial^{n-1} u}{\partial x_1 \cdots \partial x_{i-1} \partial x_{i+1} \cdots \partial x_n} +$$

$$(13) \qquad + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij} \frac{\partial^{n-2} u}{\partial x_1 \cdots \partial x_{i-1} \partial x_{i+1} \cdots \partial x_{j-1} \partial x_{j+1} \cdots \partial x_n} +$$

$$\cdots + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0 .$$

In this section we will introduce an operator notation which will greatly simplify the calculations for n^{th} order equations. This will enable us to predict the number and form of the identities, and the number and form of the invariants corresponding to each identity related to the n^{th} order equation. From this we will be able to indicate the conditions necessary to produce a cascade of equations, which will enable us to tell when a reduction in order will be possible.

In our concluding section we will discuss briefly the extensions made by Darboux, Dini, and LeRoux to systems of second order equations in one dependent variable, to a single second order equation in n indepent variables, and to a single n^{th} order equation in two independent

variables. These extensions are treated more extensively in the works of these men noted in the bibliography.

SECTION II

AN EXTENSION OF THE PRODUCT INTEGRAL TO NON-HOMOGENEOUS SYSTEMS OF THE FIRST ORDER

In Section III we shall show how the cascade method can be applied to a system of second order equations. The end result we hope to obtain is a reduction to two first order systems, one of which will be homogeneous, while the other is non-homogeneous. The solution of the homogeneous system can be found by employing Volterra's product integral[6]. As yet, however, this concept appears not to have been extended to non-homogeneous systems.[7] It will be necessary for us to do this now, to enable us to solve completely the second order systems.

Consider the system of equations

$$\frac{du_i}{dx} = \sum_{j=1}^{n} a_{ji}\, u_j + f_i\,, \qquad u_i(b) = u_{io} \qquad i = 1,2,\ldots,n, \tag{1}$$

where the a_{ji} and the f_i are given continuous, single-valued, bounded functions of the real variable x, on some non-empty interval of the real line, $b \leq x \leq c$, and the u_{io} are n given constants. First let us write (1) in the notation of matrices. We denote the row matrices

$$U(x) = (u_1\ u_2\ \ldots\ u_n)$$

$$F(x) = (f_1\ f_2\ \ldots\ f_n)$$

$$U_o = (u_{10}\ u_{20}\ \ldots\ u_{no})$$

and the square matrix

$$A = (a_{ij})_{n \times n}.$$

It will be necessary to assume that the matrix A is non-singular, that is that the equations of system (1) are linearly independent.

In the notation above, system (1) can be written as

$$(2) \qquad \frac{dU(x)}{dx} = U(x)A(x) + F(x), \qquad U(b) = U_o .$$

Let P_m be any partition of the interval $[b, c]$, such that $b = x_o < x_1 < \ldots < x_m = c$, and let ξ_ν be any point in the interval $[x_{\nu-1}, x_\nu]$. We then define U_ν by the relation

$$(3) \qquad \begin{aligned} U_\nu &= U_{\nu-1} A(\xi_\nu)(x_\nu - x_{\nu-1}) + U_{\nu-1} + F(\xi_\nu)(x_\nu - x_{\nu-1}) \\ &= U_{\nu-1}\left\{A(\xi_\nu)(x_\nu - x_{\nu-1}) + I\right\} + F(\xi_\nu)(x_\nu - x_{\nu-1}). \end{aligned}$$

Here I is the $n \times n$ identity matrix with ones on the diagonal and zeros elsewhere. From the form of this recurrence relation (3) we readily obtain that

$$(4) \qquad \begin{aligned} U_m = {} & U_o \prod_{\nu=1}^{m} \left\{A(\xi_\nu)(x_\nu - x_{\nu-1})\right\} + \\ & + \sum_{\nu=1}^{m-1} F(\xi_\nu) \left[\prod_{j=\nu+1}^{m} \left\{A(\xi_j)(x_j - x_{j-1}) + I\right\}\right] (x_\nu - x_{\nu-1}) + \\ & + F(\xi_m)(x_m - x_{m-1}). \end{aligned}$$

Next we consider a sequence of such partitions, $\{P_m\}$ such that as $m \to \infty$, $\Delta x = x_\nu - x_{\nu-1} \to 0$. Since all the functions concerned are continuous, we may proceed to the limit:

$$(5) \qquad U(c) = \lim_{m\to\infty} U_m = U_0 \prod_b^c A + \int_b^c F(\xi) \prod_\xi^c A \, d\xi .$$

In this expression, $\prod_b^c A = \prod_b^c (A(\eta)\,d\eta + I)$ is the "right" product integral of Volterra, while $\int_b^c F(\xi) \prod_\xi^c A \, d\xi$ is the row matrix of term by term Riemann integration of the elements of the row matrix, $F(\xi) \prod_\xi^c A$.

In his discussion of product integration, Schlesinger proved the following identity:[8]

$$\prod_p^q C = \prod_p^s C \prod_s^q C , \quad \text{for} \quad p < s < q.$$

Using this we see that

$$\prod_b^c A = \prod_b^\xi A \prod_\xi^c A , \qquad b < \xi < c,$$

and hence

$$(6) \qquad \prod_\xi^c A = \left(\prod_b^\xi A\right)^{-1} \prod_b^c A .$$

Putting (6) into (5) we obtain

$$(7) \qquad U(c) = U_0 \prod_b^c A + \int_b^c F(\xi) \left(\prod_b^\xi A\right)^{-1} \prod_b^c A \, d\xi .$$

Since $\prod_b^c A$ is constant with respect to the variable of integration, and since the scalar $d\xi$ commutes with every square matrix, we may factor this product integral to the right, and write (7) as

$$U(c) = \left[U_0 + \int_b^c F(\xi)\left(\prod_b^{\xi} A\right)^{-1} d\xi\right]\prod_b^c A . \tag{8}$$

This function U which we have derived is a function of the end point, c, of the interval $[b,c]$. If we vary this end point in any interval in which the functions a_{ij} and f_i remain continuous, single-valued and bounded, then $U(c)$ becomes a function of a real variable which we may call x, and hence

$$U(x) = \left[U_0 + \int_b^x F(\xi)\left(\prod_b^{\xi} A\right)^{-1} d\xi\right]\prod_b^x A . \tag{9}$$

We assert that (9) is the solution of (2) and hence, element-wise, the solution of the non-homogenous system (1). To see this, we observe first that when $x = b$, $\prod_b^x A = I$ [9], while $\int_b^x F(\xi)\left(\prod_b^x A\right)^{-1} d\xi = 0$.

Thus $U(b) = U_0$. Next we note that each row of the matrix $\prod_b^x A$ is a solution of the matrix equation $\frac{dU}{dx} = U(x)A(x)$ [10]. Therefore

$$\frac{d}{dx}\left\{\prod_b^x A\right\} = \left(\prod_b^x A\right)\cdot(A).$$

Thus differentiation of (9) yields

$$\frac{dU}{dx} = \frac{d}{dx}\left[U_0 + \int_b^x F(\xi)\left(\prod_b^{\xi} A\right)^{-1} d\xi\right]\cdot \prod_b^x A \; +$$

$$+ \left[U_0 \quad \int_b^x F(\xi)\left(\prod_b^{\xi} A\right)^{-1} d\xi\right]\cdot \frac{d}{dx}\left\{\prod_b^x A\right\} =$$

$$= F(x)\left(\prod_b^x A\right)^{-1} \prod_b^x A \; + \left[U_0 \int_b^x F(\xi)\left(\prod_b^{\xi} A\right)^{-1} d\xi\right]\left(\prod_b^x A\right) A =$$

$$= F(x) + U(x)A(x),$$

which proves that (9) is the solution of (2).

Schlesinger showed [11] that if A is a matrix of constant elements, then

$$\prod_b^x A \equiv e^{\int_b^x A d\eta}.$$

Therefore in the special case when (1) is a system with constant coefficients, (9) takes the more familiar form

$$(10) \qquad U(x) = \left[U_0 + \int_b^x F(\xi) e^{-\int_b^{\xi} A d\eta} d\xi\right] e^{\int_b^x A d\eta},$$

and this solution can be verified even more readily by direct differentiation. We note the analogy between (12) and the solution of the single non-homogeneous first order equation

$$\frac{du}{dx} = a(x)\, u(x) + f(x)$$

$$u(b) = u_o .$$

This equation has the familiar solution

$$u(x) = e^{\int_b^x a(\eta)\, d\eta} \left(u_o + \int_b^x e^{-\int_b^{\xi} a(\eta) d\eta} f(\xi)\, d\xi \right),$$

a form which we could obtain from (12) by taking the transpose.

SECTION III

SYSTEMS OF SECOND ORDER LINEAR HYPERBOLIC EQUATIONS

A. Consider the following system of second order linear hyperbolic equations:

$$(1) \quad \frac{\partial^2 u_i}{\partial x \partial y} + \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial y} + \sum_{j=1}^{n} c_{ij} u_j = 0, \quad i = 1,2,\ldots n,$$

where the coefficients a_{ij}, b_{ij}, and c_{ij} are all real functions of the real variables x and y, continuously differentiable in both variables as often as necessary, and $u_i = u_i(x,y)$ are real functions of the real variables x and y. To put this system in matrix notation[12], let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $(0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, and

we obtain

$$(2) \quad U_{xy} + AU_x + BU_y + CU = (0),$$

where $U_{xy} = \frac{\partial^2 U}{\partial x \partial y} = \begin{bmatrix} \frac{\partial^2 u_1}{\partial x \partial y} \\ \vdots \\ \frac{\partial^2 u_n}{\partial x \partial y} \end{bmatrix}$, etc. We will assume that the matrices A, B, and C are all non-singular.

We introduce the two Darboux matrix invariants

$$(3) \qquad \begin{aligned} H &= A_x + BA - C , \\ K &= B_y + AB - C . \end{aligned}$$

Then we note that (2) can be written in either of the forms

$$(4) \qquad \begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y} + AU\right) + B\left(\frac{\partial U}{\partial y} + AU\right) - HU = (0); \\ &\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x} + BU\right) + A\left(\frac{\partial U}{\partial x} + BU\right) - KU = (0). \end{aligned}$$

If we consider the substitutions

$$U_1 = U_y + AU ,$$

$$U_{-1} = U_x + BU , \qquad \text{then (4) becomes}$$

$$(5) \qquad \begin{aligned} &U_{1_x} + BU_1 - HU = (0); \\ &U_{-1_y} + AU_{-1} - KU = (0). \end{aligned}$$

If now either $H \equiv 0$ or $K \equiv 0$, where $0 = \begin{bmatrix} 0 & 0 & & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix}_{n \times n}$ our

original system (2) will be reduced to two systems of first order equations, which can be solved by the product integrals of Section II. We will make the assumption that none of our coefficient matrices are singular, except the matrix 0.

Suppose that $H \equiv 0$. Then we have the following systems to solve :

$$
(6) \qquad
\begin{aligned}
U_{1_x} + BU_1 &= (0) , \\
U_1 &= U_y + AU .
\end{aligned}
$$

The first equation of (6) has the solution

$$
(7) \qquad U_1 = \left(\prod^{x} B(\xi, y) d\xi \right)^{-1} Y^{o}(y) ,
$$

where $Y^{o}(y)$ is a column matrix of arbitrary functions of y, and $\left(\prod^{x} B(\xi, y) \, d\xi \right)^{-1}$ is the inverse of the Volterra product integral of B. Knowing U_1, we may then proceed to integrate the second equation of (6), to obtain the solution

$$
(8) \qquad U(x,y) = \left(\prod^{y} A(x, \eta) d\eta \right)^{-1} \left[X^{o}(x) + \int^{y} \left\{ \prod^{\zeta} A(x,\eta) d\eta \left(\prod^{x} B(\xi, \zeta) d\xi \right)^{-1} Y^{o}(\zeta) \right\} d\zeta \right] ,
$$

where $X^{o}(x)$ is a column matrix of arbitrary functions of x.

If $H \not\equiv 0$, but $K \equiv 0$, we can solve the system

$$
(9) \qquad
\begin{aligned}
U_{-1_y} + AU_{-1} &= (0) , \\
U_{-1} &= U_x + BU ,
\end{aligned}
$$

by the same methods utilized to solve system (6), to obtain the solution

$$(10) \qquad U(x,y) = \left(\int^x B(\xi,y)\,d\xi\right)^{-1}\left[Y^\circ(y) + \int^x \Bigg\{\int^\xi B(\xi,y)\,d\xi \left(\int^y A(\xi,\eta)\,d\eta\right)^{-1} X^\circ(\xi)\,d\xi\right].$$

Illustrative example No. 1:

Consider the system of three equations in three unknowns

$$\frac{\partial^2 u_1}{\partial x \partial y} + y\frac{\partial u_1}{\partial x} + ye^y\frac{\partial u_2}{\partial x} + y\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial y} + yu_1 + ye^y u_2 + yu_3 = 0,$$

$$\text{(a)} \quad \frac{\partial^2 u_2}{\partial x \partial y} - e^y\frac{\partial u_1}{\partial x} + y\frac{\partial u_2}{\partial x} + e^y\frac{\partial u_3}{\partial x} + \frac{\partial u_2}{\partial y} - e^y u_1 + yu_2 + e^y u_3 = 0,$$

$$\frac{\partial^2 u_3}{\partial x \partial y} + y^2\frac{\partial u_1}{\partial x} + e^y\frac{\partial u_2}{\partial x} + ye^y\frac{\partial u_3}{\partial x} + \frac{\partial u_3}{\partial y} + y^2 u_1 + e^y u_2 + ye^y u_3 = 0.$$

This system can be written in the form (2) with matrix coefficients

$$A = \begin{bmatrix} y & ye^y & y \\ -e^y & y & e^y \\ y^2 & e^y & ye^y \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = A.$$

If we compute the invariant H, we find $H \equiv 0$. Thus we may reduce to the two first order systems

$$\text{(b)} \qquad \begin{aligned} U_{1_x} + U_1 &= (0), \\ U_y + AU &= U_1. \end{aligned}$$

The first system of (b) has the immediate integral

$$U_1 = e^{-x}\, Y(y)$$

where $Y(y)$ is a column matrix of three arbitrary functions of y. Using this expression for U_1 in the second system of (b), we obtain the solution

$$U(x,y) = \left(\int^{y} A(\eta)\,d\eta\right)^{-1}\left[\,X(x) + \int^{y}\!\left(\int^{\xi} A(\eta)\,d\eta\right) e^{-x}\, Y(\xi)\,d\xi\right]$$

where $X(x)$ is a column matrix of arbitrary functions of x.

B. Consider now successively the substitutions

$$U = \Lambda U',$$

$$\begin{cases} x = \phi(x') \\ y = \psi(y') \end{cases},$$

$$\begin{cases} x = y' \\ y = x' \end{cases},$$

where Λ is an $n \times n$ matrix of twice continuously differentiable functions of x and y, such that $|\Lambda| \not\equiv 0$, and $U' = \begin{bmatrix} u_1' \\ \cdot \\ \cdot \\ \cdot \\ u_n' \end{bmatrix}$,

$u_i' = u_i'(x,y)$. Under the change of variables $U = \Lambda U'$, (2) becomes

$$\Lambda U'_{xy} + (A\Lambda + \Lambda_y)U'_x + (B\Lambda + \Lambda_x)U'_y + $$
$$+ (C\Lambda + A\Lambda_x + B\Lambda_y + \Lambda_{xy})\,U' = (0).$$

Multiplying from the left by Λ^{-1} we obtain

$$U'_{xy} + (\Lambda^{-1}A\Lambda + \Lambda^{-1}\Lambda_y)U'_x + (\Lambda^{-1}B\Lambda + \Lambda^{-1}\Lambda_x)U'_y +$$
$$+ (\Lambda^{-1}C\Lambda + \Lambda^{-1}A\Lambda_x + \Lambda^{-1}B\Lambda_y + \Lambda^{-1}\Lambda_{xy})U' = (0),$$

which is of the same form as (2):

$$(11) \quad U'_{xy} + A'U'_x + B'U'_y + C'U' = (0),$$

where
$$A' = \Lambda^{-1} A \Lambda + \Lambda^{-1} \Lambda_y ,$$

(12)
$$B' = \Lambda^{-1} B \Lambda + \Lambda^{-1} \Lambda_x ,$$
$$C' = \Lambda^{-1} C \Lambda + \Lambda^{-1} A \Lambda_x + \Lambda^{-1} B \Lambda_y + \Lambda^{-1} \Lambda_{xy} .$$

Let us compute the values of the corresponding Darboux invariants H' and K' for (11).

$$H' = A'_x + B'A' - C' =$$
$$= (\Lambda^{-1} A \Lambda + \Lambda^{-1}\Lambda_y)_x + (\Lambda^{-1} B \Lambda + \Lambda^{-1}\Lambda_x)(\Lambda^{-1} A \Lambda + \Lambda^{-1}\Lambda_y) -$$
$$-(\Lambda^{-1} C \Lambda + \Lambda^{-1} A \Lambda_x + \Lambda^{-1} B \Lambda_y + \Lambda^{-1} \Lambda_{xy}) .$$

Consider
$$\Lambda\Lambda^{-1} = I ,$$

(13)
$$\frac{\partial}{\partial x}(\Lambda\Lambda^{-1}) = \Lambda \frac{\partial \Lambda^{-1}}{\partial x} + \frac{\partial \Lambda}{\partial x}\Lambda^{-1} = \frac{\partial I}{\partial x} = 0 ,$$
$$\therefore \quad \frac{\partial \Lambda^{-1}}{\partial x} = - \Lambda^{-1} \frac{\partial \Lambda}{\partial x} \Lambda^{-1} .$$

Using this identity[13] we may differentiate the first term of H' to obtain

$$H' = -\Lambda^{-1}\Lambda_x\Lambda^{-1}A\Lambda + \Lambda^{-1}A_x\Lambda + \Lambda^{-1}A\Lambda_x - \Lambda^{-1}\Lambda_x\Lambda^{-1}\Lambda_y +$$
$$+\Lambda^{-1}\Lambda_{xy} + \Lambda^{-1}BA\Lambda + \Lambda^{-1}B\Lambda_y + \Lambda^{-1}\Lambda_x\Lambda^{-1}A\Lambda + \Lambda^{-1}\Lambda_x\Lambda^{-1}\Lambda_y -$$
$$-\Lambda^{-1}\Lambda_{xy} - \Lambda^{-1}C\Lambda - \Lambda^{-1}A\Lambda_x - \Lambda^{-1}B\Lambda_y - \Lambda^{-1}\Lambda_{xy} =$$
$$= \Lambda^{-1}(A_x + BA - C)\Lambda$$
$$= \Lambda^{-1} H \Lambda .$$

In similar manner we may compute the value of

$$K' = B'_y + A'B' - C' = \Lambda^{-1} K \Lambda .$$

We see then that this change of variables $U = \Lambda U'$ merely produces a similarity transformation on the matrices H and K, and indeed, if Λ is chosen so that $\Lambda H = H \Lambda$, then $H' = H$, and should $\Lambda K = K \Lambda$, then $K' = K$.

If we make the change of coordinates $x = \Phi(x')$, $y = \Psi(y')$, then the resulting equation

$$U_{x'y'} + \Psi'(y')AU_{x'} + \Phi'(x')BU_{y'} + \Phi'(x')\,\Psi'(y')CU = (0)$$

also has the form of (2). Here

$$A' = \Psi'(y')\,A$$
$$B' = \Phi'(x')\,B$$
$$C' = \Phi'(x')\,\Psi'(y')C,$$

and the corresponding matrix invariants are

$$H' = \Phi'(x')\,\Psi'(y')\,H,$$
$$K' = \Phi'(x')\,\Psi'(y')\,K.$$

Finally the change of coordinates $x = y'$, $y = x'$ merely has the effect of interchanging the invariants, so that $H' = K$, $K' = H$. Thus we see that the term "invariant" is correctly chosen.

C. In the likely event that neither H nor K is 0, let us consider the system

$$(14) \qquad \begin{aligned} U_1 &= U_y + AU, \\ U_{1x} + BU_1 &= HU. \end{aligned}$$

Solving the second system of (14) for U, we obtain

$$(15) \qquad U = H^{-1}U_{1x} + H^{-1}BU_1 .$$

Utilizing identity (13) we further obtain

$$(16) \qquad U_y = -H^{-1} H_y H^{-1} U_{1x} + H^{-1} U_{1xy} - H^{-1} H_y H^{-1} BU_1 + H^{-1} B_y U_1 + H^{-1} BU_{1y}.$$

Then mutliplying the first system of (14) from the left by H, and substituting (15) and (16), collecting terms we have

$$U_{1_{xy}} + A_1 U_{1_x} + B_1 U_{1_y} + C_1 U_1 = (0),$$

where

$$(17) \qquad \begin{aligned} A_1 &= (HAH^{-1} - H_y H^{-1}), \\ B_1 &= B, \\ C_1 &= (HAH^{-1} - H_y H^{-1})B + B_y - H = A_1 B_1 + B_{1y} - H. \end{aligned}$$

This equation is again of the form of (2) and we may iterate our process in an attempt to reduce this system to two systems of first order. Computing the matrix invariants for system (17) we have

$$\begin{aligned} H_1 &= A_{1x} + B_1 A_1 - C_1 \\ &= (HAH^{-1} - H_y H^{-1})_x + B(HAH^{-1} - H_y H^{-1}) - (HAH^{-1} - H_y H^{-1})B - \\ &\qquad - B_y + H \\ &= 2H - K - A_x + AB - BA + B(HAH^{-1} - H_y H^{-1}) - (HAH^{-1} - H_y H^{-1})B + \\ &\qquad + (HAH^{-1})_x - (H_y H^{-1})_x ; \end{aligned}$$

$$K_1 = H.$$

In a similar manner we may solve the system

$$\begin{aligned} U_{-1} &= U_x + BU, \\ U_{-1y} + AU_{-1} &= KU, \end{aligned}$$

to derive the system

$$(19) \qquad U_{-1_{xy}} + A_{-1}U_{-1_x} + B_{-1}U_{-1_y} + C_{-1}U_{-1} = (0)$$

where now

$$A_{-1} = A ,$$

$$B_{-1} = KBK^{-1} - K_xK^{-1},$$

$$C_{-1} = (KBK^{-1} - K_xK^{-1})A - K + A_x =$$

$$= B_{-1}A_{-1} + A_{-1_x} - K.$$

For (19) our matrix invariants will be

$$H_{-1} = K ,$$

$$K_{-1} = 2K - H - B_y + BA - AB + A(KBK^{-1} - K_xK^{-1}) - (KBK^{-1} - K_xK^{-1})A +$$

$$+ (KBK^{-1})_y - (K_xK^{-1})_y .$$

If we designate the original system of equations (2) by E, we have been able to transform E into a system of similar form, which we shall designate E_1, by the change of variables $U_1 = U_y + AU$, and into another such system, to be designated by E_{-1}, by the change of variables $U_{-1} = U_x + BU$. Provided that none of the resulting H or K invariants is non-singular, we may continue this procedure in both directions, establishing a chain of systems

$$\ldots E_{-2}, E_{-1}, E, E_1, E_2, \ldots .$$

The question naturally arises, if after making the substitution

$U_1 = U_y + AU$ and discovering that $H \not\equiv 0$, we transform system (2) into system (17), what results if we then make the substitution $U_{1-1} = U_{1_x} + B_1U_1$? Presumably we shall again obtain a system of the form of (2), but what exactly are the coefficients of this new system in relation to the original system (2) ? Utilizing the expressions for the coefficients of (19) and recalling that $K_1 \equiv H$, we obtain

$$A_{1-1} = A_1 = HAH^{-1} - H_yH^{-1},$$

$$B_{1-1} = HBH^{-1} - H_xH^{-1},$$

$$C_{1-1} = B_{1-1}A_{1-1} + A_{1_x} - H =$$

$$= H(A_x + BA - BH^{-1}H_y - AH^{-1}H_x)H^{-1} +$$

$$+ H_xH^{-1}H_yH^{-1} + H_yH^{-1}H_xH^{-1} - H_{xy}H^{-1} - H.$$

Now consider the substitution $U_{1-1} = \Lambda U'$, and for Λ let us take H, i.e. $U_{1-1} = HU'$. From (12) we see the resulting system, again of the form (2), has the coefficients

$$A' = H^{-1}A_{1-1}H + H^{-1}H_y = A,$$

$$B' = H^{-1}B_{1-1}H + H^{-1}H_x = B,$$

$$C' = H^{-1}C_{1-1}H + H^{-1}A_{1-1}H_x + H^{-1}B_{1-1}H_y + H^{-1}H_{xy} = C.$$

Thus we have transformed into a system which is exactly equivalent to system (2), since

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$H = H' = H^{-1} H_{1-1} H,$$

$$HHH^{-1} = H_{1-1},$$

hence $\qquad H = H_{1-1}$.

This result may be immediately generalized as follows: If, in system E_i we make the substitution $U_{i+1} = U_{iy} + A_i U_i$, we will obtain system E_{i+1}, but if we make the substitution $U_{i-1} = U_{ix} + B_i U_i$, we will obtain the system E_{i-1}, for all $i = \ldots, -2, -1, 0, 1, 2, \ldots$. The foregoing tacitly assumes, of course, that the corresponding H_i and K_i are neither equal to 0.

Remark #1: We may define an exponential matrix e^X, where X is any $n \times n$ matrix, by the formula

$$e^X = I + X + \frac{X^2}{2} + \frac{X^3}{3!} + \ldots = I + \sum_{i=1}^{\infty} \frac{X^i}{i!} .$$

Now unless the matrix X commutes with its x - derivative, we cannot assert that $\frac{d}{dx} e^X = e^X \frac{dX}{dx}$, since

$$\frac{d}{dx} e^X = I + \frac{1}{2}\left(\frac{dX}{dx} X + X\frac{dX}{dx}\right) + \frac{1}{3!}\left(\frac{dX}{dx} X^2 + X \frac{dX}{dx} X + X^2 \frac{dX}{dx}\right) \quad \ldots .$$

But if $X = Ay$, say, where A is an $n \times n$ matrix of constants, and y is a scalar matrix, then $\frac{dX}{dx} = A \frac{dy}{dx}$, hence

$$X\frac{dX}{dx} = Ay A\frac{dy}{dx} = A\frac{dy}{dx} \times Ay = \frac{dX}{dx} X, \text{ and thus } \frac{d}{dx}(e^X) = e^X \frac{dX}{dx} .$$

Then consider the special case of system (6), when the matrices A and B are both matrices of constants. An integrating factor for the first system of (6) would be $e^{\int B dx} = e^{Bx}$. Then we may write

$$H = H' = H^{-1} H_{l-1} H,$$

$$HHH^{-1} = H_{l-1},$$

hence $$H = H_{l-1}.$$

This result may be immediately generalized as follows: If, in system E_i we make the substitution $U_{i+1} = U_{iy} + A_i U_i$, we will obtain system E_{i+1}, but if we make the substitution $U_{i-1} = U_{ix} + B_i U_i$, we will obtain the system E_{i-1}, for all $i = \ldots, -2, -1, 0, 1, 2, \ldots$. The foregoing tacitly assumes, of course, that the corresponding H_i and K_i are neither equal to 0.

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Then consider the special case of system (6), when the matrices A and B are both matrices of constants. An integrating factor for the first system of (6) would be $e^{\int B dx} = e^{Bx}$. Then we may write

$$e^{Bx} U_{1_x} + e^{Bx} BU_1 = (0),$$

$$\frac{d}{dx}(e^{Bx} U_1) = (0), \qquad \therefore \qquad e^{Bx} U_1 = Y(y),$$

where $Y(y)$ is a column matrix of arbitrary functions of y.

Substituting $U_1 = e^{-Bx} Y(y)$ into the second equation of (6) gives

$$e^{-Bx} Y(y) = U_y + AU .$$

An integrating factor for this system is then $e^{\int A dy} = e^{Ay}$.

Integrating this system gives the solution

$$U = e^{-Ay}\left[\int e^{Ay - Bx} Y(y)dy + X(x)\right] =$$

$$= e^{-(Ay + Bx)}\left[\int e^{Ay} Y(y)dy + X(x)\right].$$

where $X(x)$ is a column matrix of arbitrary functions of x.

Now consider system (14), where no longer is $H \equiv 0$, and assume once more that A and B are constant matrices. We may integrate the first system of (14) in the above manner to obtain

$$U = e^{-Ay}\left[\int e^{Ay} U_1 dy + X(x)\right].$$

Substituting this expression for U in the second system of (14) leads to the equation

$$U_{1_x} + BU_1 = H e^{-Ay}\left[\int e^{Ay} U_1 dy + X(x)\right].$$

Multiplying from the left by $e^{Ay} H^{-1}$, this becomes

$$e^{Ay} H^{-1}(U_{1_x} + BU_1) = \int e^{Ay} U_1 dy + X(x) .$$

If we now differentiate both sides with respect to y we obtain

$$(e^{Ay}AH^{-1} - e^{Ay}H^{-1}H_yH^{-1})(U_{1_x} + BU_1) +$$

$$+ e^{Ay}H^{-1}(U_{1_{xy}} + BU_{1_y} + B_yU_1) = e^{Ay}U_1 .$$

Multiplying from the left by He^{-Ay}, and collecting terms we finally obtain

$$U_{1_{xy}} + A_1 U_{1_x} + B_1 U_{1_y} + C_1 U_1 = (0)$$

where the coefficients A_1, B_1, and C_1 are exactly as obtained for (17).

The curious thing about this second method for starting a chain of equations is that even if A and B do not commute with their derivatives, we may still use the integrating factor $e^{\int A dy}$ and operating formally as above, pretending that the equation $\frac{d}{dy} e^{\int A dy} = e^{\int A dy}.A$ were actually true, we will arrive once more at system (17) with the correct coefficients A_1, B_1, and C_1. Thus we obtain a valid equation from an invalid operation, but an operation which appears valid on the surface, and is completely analogous to the operations performed on a single equation of this type.

D. If the chain has been continued to the $i + 1^{st}$ system of equations, we can readily establish the following recurrence relationships for the matrix invariants:

(20) $$H_{i+1} = 2H_i - K_i - A_{i_x} + A_iB_i - B_iA_i + B_i(H_iA_iH_i^{-1} - H_{i_y}H_i^{-1}) -$$
$$-(H_iA_iH_i^{-1} - H_{i_y}H_i^{-1})B_i + (H_iA_iH_i^{-1})_x - (H_{i_y}H^{-1})_x ;$$

$$K_{i+1} = H_i ,$$

$$i = \ldots -2, -1, 0, 1, 2, \ldots .$$

These may be solved for H_i and K_i to give

(21) $$H_i = K_{i+1},$$

$$K_i = 2K_{i+1} - H_{i+1} - A_{i_x} + A_iB_i - B_iA_i + B_i(K_{i+1}A_iK_{i+1}^{-1} - K_{i+1_y}K_{i+1}^{-1}) -$$
$$- (K_{i+1}A_iK_{i+1}^{-1} - K_{i+1_y}K_{i+1}^{-1})B_i + (K_{i+1}A_iK_{i+1}^{-1})_x - (K_{i+1_y}K_{i+1}^{-1})_x.$$
$$i = \ldots, -2, -1, 0, 1, 2, \ldots$$

From these we immediately obtain the relations

(22) $$H_{i+1} + H_{i-1} = 2H_i - A_{i_x} + A_iB_i - B_iA_i + B_i(H_iA_iH_i^{-1} - H_{i_y}H_i^{-1}) -$$
$$-(H_iA_iH_i^{-1} - H_{i_y}H_i^{-1})B_i + (H_iA_iH_i^{-1})_x - (H_{i_y}H_i^{-1})_x ;$$

(23) $$H_{i+1} = H_i + H - K + \sum_{j=0}^{i}\Big\{A_jB_j - B_jA_j - A_{jx} + B_j(H_jA_jH_j^{-1} - H_{j_y}H_j^{-1}) -$$
$$-(H_jA_jH_j^{-1} - H_{j_y}H_j^{-1})B_j + (H_jA_jH_j^{-1})_x - (H_{j_y}H_j^{-1})_x .$$
$$i = \ldots -2, -1, 0, 1, 2, \ldots$$

Finally (5) may now be written more generally as

(24) $$U_{i_x} + BU_i = H_{i-1}U_{i-1},$$
$$U_{-i_y} + AU_{-i} = K_{-(i-1)}U_{-(i-1)}.$$

Considering the first system of (24), suppose B is a matrix of constant elements. Multiplying both sides by the exponential matrix e^{Bx} (see remark 1.) and solving for U_{i-1}, we obtain

$$U_{i-1} = H_{i-1}^{-1}\, e^{-Bx}\frac{\partial}{\partial x}(e^{Bx}U_i).$$

This immediately leads to an expression for U in terms of U_i and its derivatives with respect to x, namely

(25) $$U = H^{-1}e^{-Bx}\frac{\partial}{\partial x}\Big(e^{Bx}H_1^{-1}e^{-Bx}\frac{\partial}{\partial x}\Big(e^{Bx}H_2^{-1}e^{-Bx}\frac{\partial}{\partial x}\Big(\ldots\frac{\partial}{\partial x}(e^{Bx}U_i)\Big)\ldots\Big),$$

In the case where the matrix B is such that $\frac{d}{dx}(e^{\int B dx}) = Be^{\int B dx}$ does not hold, the solution for U in terms of U_i becomes much more involved. Solving the first equation of (24) for U_{i-1}, we have $U_{i-1} = H_{i-1}^{-1}(U_{i_x} + BU_i)$. Then $U_{i-2} = H_{i-2}^{-1}(U_{i-1_x} + BU_{i-1}) = H_{i-2}^{-1}\left(\frac{\partial}{\partial x}(H_{i-1}^{-1}(U_{i_x} + BU_i) + BH_{i-1}^{-1} \cdot (U_{i_x} + BU_i)\right)$. Iterating this process we see that eventually we will obtain $U_{i-i} = U$ in terms of U_i and the i invariants $H, H_1, \ldots H_{i-1}$. Using the relation

$$U_i = U_{i-1_y} + A_{i-1} U_{i-1},$$

we may, in a similar manner, obtain an expression for U_i, in terms of U.

E. We shall now give consideration to what may occur in the form of the invariants as we iterate along the chain of systems. Perhaps the most natural point to investigate is that of "periodic" systems, i.e. systems such that after j iterations, we obtain $E_j = E$. We shall say that such systems have period j. Consider first a system of period 1. Then $E_1 = E$, $H_1 = H$, and $K_1 = K$. Hence from (20) we have $H = K$, and

$$H = 2H - H - A_x + AB - BA + B(HAH^{-1} - H_yH^{-1}) - (HAH^{-1} - H_yH^{-1})B + $$
$$+ (HAH^{-1})_x - (H_yH^{-1})_x.$$

In the special case where $AB = BA$ and $B(HAH^{-1} - H_yH^{-1}) = (HAH^{-1} - H_yH^{-1})B$, this reduces to

$$(HAH^{-1})_x - (H_yH^{-1})_x - A_x = 0,$$

which has the immediate integral

$$HAH^{-1} - H_yH^{-1} - A = -Y(y), \tag{26}$$

where $Y(y)$ is an $n \times n$ matrix of arbitrary functions of y.

If in addition $HA = AH$, (26) is further reduced to'

$$H_yH^{-1} = Y(y)$$
$$H_y = Y(y)\,H\,.$$

This may be further integrated by the product integral to give

$$H = \left(\prod^{y} Y d\eta\right) X(x), \tag{27}$$

where $X(x)$ is an $n \times n$ matrix of arbitrary functions of x. If we select for $Y(y)$ a matrix of constant elements, D this solution can be written

$$H = e^{Dy}\,X(x)\,. \tag{28}$$

Suppose we have a system of period one and let us make a change of variables $U = \Lambda U'$. This will not change the periodicity or the equality of the invariants, for, as we have seen, if $H = K$, then

$$H' = \Lambda^{-1} H \Lambda = \Lambda^{-1} K \Lambda = K'.$$

Let us select Λ however, so that our coefficient $A' = 0$. This simply requires that $\Lambda_y = -A\Lambda$, and hence

$$\Lambda = \prod^{y} (-A) d\eta \quad X^*(x), \tag{29}$$

where $X^*(x)$ is a column matrix of arbitrary functions of x.

Then $H' = A'_x + B'A' - C' = -C' = K'$, but $K' = B'_y + A'B' - C' = B'_y - C'$, hence $B'_y = 0$. If we make further assumptions regarding the character of A and B, our system can be reduced further. Let A and B be matrices of constant elements. Then (29) can be expressed as

(30) $$\Lambda = e^{-Ay} X^*(x).$$

Since $B' = \Lambda^{-1} B \Lambda + \Lambda^{-1}\Lambda_x$, we may also attempt to find Λ such that $B' = 0$. This gives

(31) $$\Lambda = e^{-Bx} Y^*(y)$$

Comparing (30) and (31) and noting that $e^P e^Q = e^{P+Q}$ if and only if $PQ = QP$,[14] we may take for our arbitrary matrices $X^* = e^{-Bx}$, $Y^* = e^{-Ay}$ to get the matrix

(32) $$\Lambda = e^{-(A_y + B_x)}$$

if and only if $AB = BA$.

Using (32) to change coordinates under $U = \Lambda U'$, our resulting coefficients are

$$A' = 0, \quad B' = 0, \quad C' = -H',$$

and our system (11) has the reduced form

(33) $$U'_{xy} = H'U',$$

which is the matrix analog to the telegraph equations. Since, if A and B are constants such that $BA = AB$, it follows that $H = K$, we have proved the following:

<u>Theorem I. A system of equations of the form (2) having constant matrix coefficients A and B can be reduced to the form $U'_{xy} = H'U'$ by a change of variables $U = \Lambda U'$ if and only if $AB = BA$.</u>

Illustrative example #2:

$$\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial u_1}{\partial x} + \frac{3\partial u_2}{\partial x} + \frac{5\partial u_3}{\partial x} + \frac{7\partial u_1}{\partial y} + \frac{2\partial u_2}{\partial y} + 28\frac{\partial u_3}{\partial y} + 10u_1 + 45u_2 + 149u_3 = 0,$$

$$\text{(c)} \quad \frac{\partial^2 u_2}{\partial x \partial y} + \frac{2\partial u_1}{\partial x} - \frac{2\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} - \frac{2\partial u_1}{\partial y} + \frac{11\partial u_2}{\partial y} + \frac{12\partial u_3}{\partial y} + 20u_1 - 17u_2 + 49u_3 = 0,$$

$$\frac{\partial^2 u_3}{\partial x \partial y} + \frac{\partial u_2}{\partial x} + \frac{4\partial u_3}{\partial x} + \frac{2\partial u_1}{\partial y} + \frac{2\partial u_2}{\partial y} + \frac{17\partial u_3}{\partial y} + 6u_1 + 19u_2 + 79u_3 = 0\,.$$

This system may be written in the form (2), with the constant matrix coefficients

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 2 & 28 \\ -2 & 11 & 12 \\ 2 & 2 & 17 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 45 & 149 \\ 20 & -17 & 49 \\ 6 & 19 & 79 \end{bmatrix}.$$

We observe first that det $A = -23$, det $B = 529$, and det $C = -8,792$, so that all matrices concerned are non-singular. (Notice that $B = A^2$ and $C = A^3 - I$.) Computing the values of H and K we obtain

$$H = A_x + BA - C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 11 & 45 & 149 \\ 20 & -16 & 49 \\ 6 & 19 & 80 \end{bmatrix} - \begin{bmatrix} 10 & 45 & 149 \\ 20 & -17 & 49 \\ 6 & 19 & 79 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$$K = B_y + AB - C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 11 & 45 & 149 \\ 20 & -16 & 49 \\ 6 & 19 & 80 \end{bmatrix} - \begin{bmatrix} 10 & 45 & 149 \\ 20 & -17 & 49 \\ 6 & 19 & 79 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

Thus $H = K = I$, and we further observe that $AB = BA$. The conditions of theorem I being satisfied, (c) reduces to

(d) $$U'_{xy} = H'U'.$$

Under the transformation $U = \Lambda U'$, however, we know that $H' = \Lambda^{-1} H \Lambda = \Lambda^{-1} I \Lambda = \Lambda^{-1}\Lambda = I = H$, so that (d) is in reality a system of three "uncoupled" telegraph equations,

(e) $$U'_{xy} = U'.$$

Each equation of this system has the same solution u', which can be found in the literature.[15] Thus (e) has the solution $U' = \begin{bmatrix} u' \\ u' \\ u' \end{bmatrix}$.

Having determined U', we must then compute the solution U, using the relation $U = \Lambda U'$. Equation (32) tells us that

$$\Lambda = e^{-(Ay+Bx)} = e^{-y\begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & 1 \\ 0 & 1 & 4 \end{bmatrix} - x\begin{bmatrix} 7 & 2 & 28 \\ -2 & 11 & 12 \\ 2 & 2 & 17 \end{bmatrix}},$$

hence our solution is

$$U = e^{-(Ax + By)} \, U_1 .$$

F. We consider next the systems of period two, so that $E_2 = E$, and hence $H_2 = H$ and $K_2 = K$. Equations (20) and (22) tell us that

$$(34) \qquad 2H = 2K - A_{1x} + A_1B_1 - B_1A_1 + B_1(KA_1K^{-1} - K_yK^{-1}) - (KA_1K^{-1} - K_yK^{-1})B_1 + (KA_1K^{-1})_x - (K_yK^{-1})_x ,$$

and

$$(35) \qquad 2K = 2H - A_x + AB - BA + B(HAH^{-1} - H_yH^{-1}) - (HAH^{-1} - H_yH^{-1})B + (HAH^{-1})_x - (H_yH^{-1})_x .$$

Summing these two equations we obtain

$$(36) \qquad \left\{(KA_1K^{-1})_x - A_{1x}\right\} + \left\{(HAH^{-1})_x - A_x\right\} + \left\{A_1B_1 - B_1A_1\right\} + \left\{AB - BA\right\} + \left\{B_1(KA_1K^{-1} - K_yK^{-1}) - (KA_1K^{-1} - K_yK^{-1})B_1\right\} + \left\{B(HAH^{-1} - H_yH^{-1}) - (HAH^{-1} - H_yH^{-1})B\right\} - (K_yK^{-1})_x - (H_yH^{-1})_x = 0 .$$

Looking at the terms within each pair of curly brackets, we note that under certain obvious conditions of commutativity this system reduces to

$$(37) \qquad (K_yK^{-1})_x + (H_yH^{-1})_x = 0 .$$

Equation (37) has the immediate integral

$$(38) \qquad K_yK^{-1} + H_yH^{-1} = Y(y)$$

[illegible]

where $Y(y)$ is a square matrix of functions of y.

If we impose another condition of commutativity, whereby $KH = HK$, $K_yH = HK_y$ then an integrating factor will be multiplication from the right by KH. For this leads to

$$K_yH + H_yK = Y(HK),$$

$$\frac{\partial}{\partial y}(HK) = Y(HK).$$

Hence

(39) $$HK = \left\{ \int^y (Y(\eta)\, d\eta \right\} X(x),$$

where $X(x)$ is a square matrix of functions of x. These functions of x and y are in fact determined by the coefficients A, B, and C. Thus we have proved Theorem II:

If a system of equations of the form (2) has period two, and the following conditions of cummutativity are satisfied:

(a) $KA_1 = A_1K$

(b) $HA = AH$

(c) $AB = BA$

(d) $A_1B_1 = B_1A_1$

(e) $B_1K_yK^{-1} = K_yK^{-1}B_1$

(f) $BH_yH^{-1} = H_yH^{-1}B$

(g) $KH = HK$

(h) $K_yH = HK_y$

then the product of the two matrix invariants H and K has the form

$$HK = Y'(y) \quad X'(x),$$

where $Y'(y)$ and $X'(x)$ are square matrices of functions of y and x respectively.

Since H is non-singular, there exists a matrix Θ defined by the relation $H = e^{\Theta}$, so that $\Theta = \log H$.[16] Now if the conditions of theorem II are satisfied; and in addition, (39) takes the simplified form $HK = I$, then (34) can be written

$$2H = 2K - (K_y K^{-1})_x,$$

or

$$2H - 2H^{-1} = (H^{-1}H_y)_x,$$

hence

(40) $$2e^{\Theta} - 2e^{-\Theta} = (H^{-1}H_y)_x.$$

Furthermore, if Θ has the property that $\Theta \dfrac{\partial\Theta}{\partial y} = \dfrac{\partial\Theta}{\partial y}\Theta$, then $H_y = e^{\Theta}\dfrac{\partial\Theta}{\partial y} = H\dfrac{\partial\Theta}{\partial y}$, so that

(41) $$\frac{\partial\Theta}{\partial y} = H^{-1}H_y.$$

Substituting (41) into (40) yields the equation

(42) $$\frac{\partial^2\Theta}{\partial x\partial y} = 2e^{\Theta} - 2e^{-\Theta},$$

and this non-linear system of equations will then have the solution $\Theta = \log H$. Equation (42) is the analog of Liouville's equation.[17]

G. We come now to a consideration of the general form of the solution when some $H_i = 0$ $H_j \neq 0$, $0 \leq j < i$. That is, we iterate the process forming the chain of equations $E, E_1, E_2, \dots E_i$, until $H_i = 0$. It will then be impossible to form equation E_{i+1}, hence E_i is the first system for which the H invariant vanishes. The first equation of (4) then has the form

$$\frac{\partial}{\partial x}\left(\frac{\partial U_i}{\partial y} + A_i U_i\right) + B_i \left(\frac{\partial U_i}{\partial y} + A_i U_i\right) = (0).$$

This has the immediate first integral

$$\left(\frac{\partial U_i}{\partial y} + A_i U_i\right) = \left(\int^x B_i d\eta\right)^{-1} Y(y),$$

where $Y(y)$ is a column matrix of functions of y only. Upon integrating a second time, we obtain the solution

$$\text{(43)}\quad \begin{aligned} U_i(x,y) = &\left(\int^y A_i(x,\eta)\, d\right)^{-1} \Big[X(x) + \\ &+ \int^y \left\{ \int^{\zeta} A_i(x,\eta)\, d\eta \left(\int^x B_i(\xi,\zeta)\, d\xi\right)^{-1} Y(\zeta) \right\} d\zeta\Big]. \end{aligned}$$

where $X(x)$ is a column matrix of functions of x only.

This solution is of the form

$$U_i = \mathcal{A}\left[X(x) + \int^y \mathcal{B}\, Y(y)\, dy\right],$$

where $\mathcal{A}$ and $\mathcal{B}$ are $n \times n$ matrices of certain well defined functions of x and y.

Using the iteration scheme of paragraph D, we see that the solution for U has the form

$$\text{(44)}\quad \begin{aligned} U = &\mathcal{A}\left(X + \int \mathcal{B}\, Y dy\right) + \mathcal{A}_1\left(X' + \int \frac{\partial \mathcal{B}}{\partial x} Y dy\right) + \\ &+ \mathcal{A}_2\left(X'' + \int \frac{\partial^2 \mathcal{B}}{\partial x^2} Y dy\right) + \dots + \mathcal{A}_i\left(X^{(i)} + \int \frac{\partial^i \mathcal{B}}{\partial x^i} Y dy\right). \end{aligned}$$

Here $\mathcal{Q}, \mathcal{Q}_1, \ldots, \mathcal{Q}_i$ designate certain well defined $n \times n$ matrices of functions of x and y, and are composed of certain products of H_j^{-1}, $\mathcal{Q}_j$, B, and their derivatives. We note that the column matrix $\mathbf{Y}$ always appears under the integral sign, in the most general case. If the boundary conditions are such that $Y \equiv (0)$ we see that the solution is of the form

$$(45) \qquad U = \mathcal{Q}X + \mathcal{Q}_1X' + \mathcal{Q}_2X'' + \ldots + \mathcal{Q}_iX^{(i)}, \ \mathcal{Q}_i \neq 0,$$

and this is the most general solution in which appear arbitrary functions of x, and no integral signs.

Conversely, if the original system of equations has a particular solution of the form of (45), then the repeated application of the cascade method will certainly lead, after a number of iterations not greater than i, to a system for which $H_i \equiv 0$. We will now prove this, i.e. if (45) is a particular solution, the number of iterations necessary before any H-invariant vanishes, is not greater than i. If we substitute (45) into the system (2), we will obtain an expression of the form

$$\mathcal{H}X + \mathcal{H}_1X' + \ldots + \mathcal{H}_{i+1}X^{(i+1)} = (0).$$

From the arbitrary character of X, we assert that

$$(46) \qquad \mathcal{H}_j = 0 \qquad j = 0, 1, 2, \ldots, i+1.$$

If we compute the form of $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ we obtain

$$\mathcal{H}_i = \frac{\partial \mathcal{Q}_{i-1}}{\partial y} + A\mathcal{Q}_{i-1} \frac{\partial^2 \mathcal{Q}_i}{\partial x \partial y} + A\frac{\partial \mathcal{Q}_i}{\partial x} + B\frac{\partial \mathcal{Q}_i}{\partial y} + C\mathcal{Q}_i,$$

$$\mathcal{H}_{i+1} = \frac{\partial \mathcal{Q}_i}{\partial y} + A\mathcal{Q}_i .$$

Equation (46) implies that

(47) $$\frac{\partial a_i}{\partial y} + A a_i = 0,$$

thus $$\mathcal{X}_1 = \frac{\partial a_{i-1}}{\partial y} + A a_{i-1} + \frac{\partial}{\partial x}\left(\frac{\partial a_i}{\partial y} + A a_i\right) + B\left(\frac{\partial a_i}{\partial y} + A a_i\right) -$$

$$-\left(A_x + BA - C\right) a_i = \frac{\partial a_{i-1}}{\partial y} + A a_{i-1} - H a_i = 0.$$

Hence we have that

(48) $$\frac{\partial a_{i-1}}{\partial y} + A a_{i-1} = H a_i .$$

Recall now the first substitution

(49) $$U_1 = \frac{\partial U}{\partial y} + AU .$$

If we substitute the expression for U given by (45) into (49), and utilize (47), we see immediately that the resulting expression for U_1 will have no derivatives of order greater than i - 1. In addition, (48) informs us that the coefficient of $X^{(i-1)}$ in the expression for U_1 will be $H a_i$. Whenever H vanishes, therefore, the coefficient of $X^{(i-1)}$ in this expression vanishes also. That is to say, if H vanishes the expression for U_1 would have no derivatives of X of order greater than i-2. As a consequence, repeated application may lead to a system E_j, for $j < i$, for which the invariant $H_j = 0$; otherwise we will obtain an equation E_i, which has a particular solution of the form

$$U_i = a X .$$

To summarize, we have proved the <u>theorem III:</u>

[illegible]

[illegible]

[illegible]

[illegible]

<u>If system (2) has a particular integral of the form (45), with $\mathcal{A}_i \neq 0$ then the method of Laplace will lead, after a number of iterations not greater than i, to an equation which is integrable.</u>

H. Consider now expressions of the form

$$U = \mathcal{A} X + \mathcal{A}_1 X' + \ldots + \mathcal{A}_i X^{(i)},$$

which contain a matrix of arbitrary functions of x, and derivatives of this matrix up to a specified order. It is obviously quite often possible to express U in terms of derivatives of an order greater than specified. That is, if X is expressible as a sum of matrices:

$$X = \mathcal{B} X_1 + \mathcal{C} X_1' + \ldots + \mathcal{L} X^{(\nu)},$$

where $\mathcal{B}, \mathcal{C}, \ldots \mathcal{L}$ are certain matrix functions of x, and X_1 is a new matrix of arbitrary functions of x; then U may contain derivatives of X_1 up to order $i + \nu$. Conversely, it may be possible to reduce the order of the highest derivative appearing in U. For instance, if

$$U = \mathcal{A}(X + X') + \mathcal{B}(X' + X''),$$

then the substitution $X_1 = X + X'$ will reduce the order of the highest derivative by 1.

We will say that the matrix U has <u>rank $i+1$</u> with respect to x if the highest derivative of X appearing in the expression for U is of order i, and it is <u>impossible to reduce</u> this order by substitutions of the type described above.

We assert, that if the system (E_i) is the first for which the

[illegible]

[illegible]

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invariant H_i vanishes, then there exists a particular solution of the original system, composed of a column matrix X of arbitrary functions of x, and derivatives of this matrix up to and including order i. This solution is irreducible in the order of derivatives, and hence has the rank $i+1$.

For if there were a substitution which would reduce the order of the highest derivatives to $i-\nu$, say, then there would be a system (E_j) $j = i-\nu < i$, for which this new expression would be a particular integral. But this would imply that $H_j = 0$, which is contrary to our hypothesis. As a result, it is evident that if (E_i) is the first system for which $H_i = 0$, then no other system (E_{i-k}) with positive or negative indices will admit a solution of rank $i+1$ with respect to x.

The results of the preceding discussion apply without modification to the second substitution, i.e. the substitution which results in the chain (E_{-1}) $(E_{-2})\ldots$ If this chain eventually terminates at some system (E_{-j}) for which $K_{-j} = 0$, then there exists a particular integral of rank $j+1$ with respect to y,

$$U = \mathcal{B}Y + \mathcal{B}_1 Y' + \ldots + \mathcal{B}_j Y^{(j)},$$

and conversely.

I. We are now in a position to construct all the systems of a given dimension (where by the dimension is meant the number of dependent variables) of the form (2), which will lead to a general integral by this extension of Laplace's cascade method. Suppose, for example, we wish to

[illegible]

construct a system of dimension n, which will have a solution of rank $i+1$ with respect to x. Then the chain must terminate after i operations, so that the system (E_i) will be integrable. First we choose arbitrarily matrices A_i and B_i non-singular, of order n. Then the equation.

$$(50) \qquad H_i = A_{i_x} + B_i A_i - C_i = (0)$$

will determine C_i. Then a particular integral is given by (43). The value of the K_i invariant for (E_i) is determined by

$$(51) \qquad K_i = B_{i_y} + A_i B_i + C_i,$$

and then the relations (17) and (21) will permit us to calculate the invariants for the systems $(E_{i-1}), \ldots, (E)$,

Viz:

$$H_{i-1} = K_i,$$

$$B_{i-1} = B_i,$$

$$A_{i-1} = H_{i-1}^{-1} (A_i (+ H_{i-1_y} H_{i-1}^{-1}) H_{i-1}$$

$$(52) \qquad = H_{i-1}^{-1} A_i H_{i-1} + H_{i-1}^{-1} H_{i-1_y},$$

$$K_{i-1} = 2K_i - H_i - A_{i_x} + A_{i-1}B_{i-1} - B_{i-1}A_{i-1}$$

$$+ B_{i-1}(K_i A_{i-1} K_i^{-1} - K_{i_y} K_i^{-1}) - (K_i A_{i-1} K_i^{-1} - K_{i_y} K_i^{-1}) B_{i-1} +$$

$$+ K_i A_{i-1} K_i^{-1})_x - (K_{i_y} K_i^{-1})_x .$$

These relations also permit the calculation of the coefficients A, B, and C. From the expression in (52) for A_{i-1} and B_{i-1}, we can show that

$$A = (H_{i-1} H_{i-2} \cdots H_1 H)^{-1} A_i (H_{i-1} H_{i-2} \cdots H_1 H) +$$

$$+ \sum_{n=1}^{i-1} \left(\prod_{j=n}^{i} H_{i-j} \right)^{-1} \left(H_{i-n} \right)_y \left(\prod_{j=n+1}^{i} H_{i-j} \right) + H^{-1} H_y ,$$

$$B = B_i ,$$

and, having determined K,

$$C = B_y + AB - K .$$

Having thus determined all of the invariant matrices H_j, $0 \leq j < i$, we may then compute the general integral for U by the iteration process of paragraph D.

We may determine in a similar manner all systems which terminate in both senses, and hence admit a particular solution of the form

$$U = \mathcal{A} X + \mathcal{A}_1 X' + \ldots + \mathcal{A}_i X^{(i)} + \mathcal{B} Y + \ldots \mathcal{B}_j Y^{(j)}$$

which contains no integral sign. The preceding expression has rank $i + 1$ with respect to x and rank $j + 1$ with respect to y. The sum $i + j$ is called the <u>characteristic number</u> of the equation. This number does not change upon successive applications of the Laplace Method. In passing from system (E) to system (E_h) for example, the number i is diminished by h, but the number j is increased by the same amount; the sum of the two is unchanged. This is apparent if we consider the system (E_i), with invariant $H_i = 0$. This is of the form

(53) $$\frac{\partial}{\partial x}\left(\frac{\partial U_i}{\partial y} + A_i U_i\right) + B_i\left(\frac{\partial U_i}{\partial y} + A_i U_i\right) = (0),$$

which admits a solution of rank 1 with respect to x, and of rank $i + j + 1$ with respect to y. For brevity, let $n = i + j$ and consider the substitution

$$U_i = \left(\int A_i dy\right)^{-1} \Theta ,$$

where Θ is a column matrix of unknown functions.

Then $$\frac{\partial U_i}{\partial y} = \left(\int A_i dy\right)^{-1} \frac{\partial \Theta}{\partial y} - A_i \cdot \left(\int A_i dy\right)^{-1} \Theta ,$$

hence $$\frac{\partial U_i}{\partial y} + A_i U_i = \left(\int A_i dy\right)^{-1} \frac{\partial \Theta}{\partial y} .$$

Using this (53) becomes

(54) $$\frac{\partial}{\partial x}\left\{\left(\int A_i dy\right)^{-1} \frac{\partial \Theta}{\partial y}\right\} + B_i \left\{ \left(\int A_i dy\right)^{-1} \frac{\partial \Theta}{\partial y} = (0) .$$

An integrating factor for (54) is multiplication from the left by $\int B_i dx$. Thus we may write

(55) $$\frac{\partial}{\partial x}\left\{\int B_i dx \left(\int A_i dy\right)^{-1} \frac{\partial \Theta}{\partial y}\right\} = (0) .$$

Denoting $\int B_i dx \cdot \left(\int A dy\right)^{-1} = \alpha^{-1}$, (55) becomes

$$\frac{\partial}{\partial x}\left\{\alpha^{-1} \frac{\partial \Theta}{\partial y}\right\} = (0),$$

which may be immediately integrated to give

$$(56) \qquad \frac{\partial \Theta}{\partial y} = \alpha Y_1 , \qquad \Theta = \int \alpha Y_1 dy + X_1 ,$$

where Y_1 , designates a column matrix of arbitrary functions of y, and X_1 a column matrix of arbitrary functions of x. We know, already, that (56) admits a solution of the form,

$$(57) \qquad \Theta = X_1 + \mathcal{B} Y + \mathcal{B}_1 Y^1 + \dots \mathcal{B}_n Y^{(n)},$$

in which $\mathcal{B}, \mathcal{B}_1 \dots, \mathcal{B}_n$ are certain well-defined matrices of functions of x and y, and Y is a colum matrix of functions of y only.

If we substitute (57) in (56) and take an arbitrary numerical value x, we find that Y and Y_1 are related by an expression of the form

$$(58) \qquad Y_1 = \Lambda Y + \Lambda_1 Y' + \dots + \Lambda_{n+1} Y^{(n+1)},$$

in which Λ , Λ_1, ... Λ_{n+1} are certain square matrices of functions of y only. Substituting the expressions for Θ and Y_1 given by (57) and (58) into the first equation of (56) implies that

$$(59) \qquad \frac{\partial}{\partial y}\left\{\mathcal{B}Y + \mathcal{B}_1 Y' + \dots + \mathcal{B}_n Y^{(n)}\right\} = \alpha \left(\Lambda Y + \Lambda_1 Y' + \dots + \Lambda_{n+1} Y^{(n+1)}\right).$$

In order that (59) be a valid equation, the coefficients of like order

derivatives on each side must be equal. In this manner we obtain the system

$$
\begin{aligned}
(60)\qquad & \frac{\partial \mathcal{B}}{\partial y} = \alpha\Lambda, \\
& \frac{\partial \mathcal{B}_1}{\partial y} + \mathcal{B} = \alpha\Lambda_1, \\
& \quad \cdots \\
& \frac{\partial \mathcal{B}_n}{\partial y} + \mathcal{B}_{n-1} = \alpha\Lambda_n, \\
& \mathcal{B}_n = \alpha\Lambda_{n+1}.
\end{aligned}
$$

If we eliminate the $\mathcal{B}_j$ from (60), we obtain the equation

$$
(61)\qquad \alpha\Lambda - \frac{\partial}{\partial y}(\alpha\Lambda_1) + \frac{\partial^2}{\partial y^2}(\alpha\Lambda_2) - \cdots + (-1)^{n+1}\frac{\partial^{n+1}}{\partial y^{n+1}}(\alpha\Lambda_{n+1}) = 0,
$$

a linear equation of order $n+1$ with respect to α, whose matrix coefficients contain functions of y only. If we solve (60) for the $\mathcal{B}_j$, we obtain the values

$$
\begin{aligned}
(62)\qquad & \mathcal{B}_n = \alpha\Lambda_{n+1}, \\
& \mathcal{B}_{n-1} = \alpha\Lambda_n - \frac{\partial}{\partial y}(\alpha\Lambda_{n+1}), \\
& \quad \cdots \\
& \mathcal{B} = \alpha\Lambda_1 - \frac{\partial}{\partial y}(\alpha\Lambda_2) + \cdots + (-1)^n \frac{\partial^n}{\partial y^n}(\alpha\Lambda_{n+1}),
\end{aligned}
$$

which permits the determination of the expression for Θ.

The relation (58), between the matrices Y and Y_1, permits an arbitrary choice of either one or the other for the definition of the value of Θ, by either (57) or the second equation

of (56). The choice of the former, however, will give the most general form of the integral for Θ .

As an application of the preceding discussion, which is complementary to our original problem, we see that it is sufficient to choose $n+2$ square matrices of arbitrary functions of y, namely $\Lambda, \Lambda_1, \ldots, \Lambda_{n+1}$, solve the resulting linear equation of $n+1^{st}$ order in one independent variable, (61), for the matrix α , in order to determine the form of the equation (E_1). The formulae (62) and (57) then provide the general integral of the equation (56) and hence (53). Repeated application of the substitutions of Laplace will enable us to determine the system (E) which has this general integral.

Now consider an arbitrary linear equation of $(n+1)^{st}$ order of the form

$$(63) \qquad M\alpha + M_1\alpha' + \ldots + M_{n+1}\alpha^{(n+1)} = 0,$$

where $M, M_1 \ldots, M_{n+1}$ are any $n+2$ square matrices of known functions y and α is a square matrix of functions of y . Let $\Lambda, \Lambda_1, \ldots \Lambda_{n+1}$ be defined by the identity

$$(64) \qquad \Lambda W + \Lambda_1 W' + \Lambda_2 W'' + \ldots + \Lambda_{n+1} W^{(n+1)} =$$
$$= MW - \frac{\partial}{\partial y}(M_1 W) + \frac{\partial^2}{\partial y^2}(M_2 W) - \ldots + (-1)^{n+1}\frac{\partial^{n+1}}{\partial y^{n+1}}(M_{n+1}W),$$

where W is any arbitrary square matrix of functions of y. (64) then determines the Λ's in terms of the M's, for example

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$$(65)\qquad \Lambda = M - \frac{\partial M_1}{\partial y} + \frac{\partial^2 M_2}{\partial y^2} - \dots + (-1)^{n+1} \frac{\partial^{n+1} M_{n+1}}{\partial y^{n+1}},$$
$$\Lambda_{n+1} = (-1)^{n+1} M_{n+1}.$$

In light of equations (60) and (65) we may set

$$M_{n+1} = (-1)^{n+1} \cdot \alpha^{-1} \mathcal{B}_n.$$

That is, if α is a solution of (63), then we may determine the functions Λ_j by the identity (64), and the $\mathcal{B}_j$ by the system (60). Upon substitution into (57), we see that the general integral contains the column matrix Y and its derivatives up to order n. Thus we have proved the following:

<u>Theorem IV: In order that the linear equation</u>

$$\frac{\partial}{\partial x}\left\{\alpha^{-1} \frac{\partial \Theta}{\partial y}\right\} = (0)$$

<u>admit a solution of rank $n+1$ with respect to y, that is for the general solution to be of the form</u>

$$\Theta = X + \mathcal{B} Y + \mathcal{B}_1 Y' + \dots + \mathcal{B}_n Y^{(n)},$$

<u>such that the solution cannot be written in analogous form in which there appear fewer derivatives of the arbitrary column matrix Y, it is necessary and sufficient that α, considered as a function of y, satisfy a linear equation of order $n+1$ whose coefficients are functions of y, and that α does not satisfy a similar linear equation of lesser order.</u>

J. The results given in the section have been kept in close analogy to those contained in Chapter II, volume 2 of <u>Lecons Sur La Theorie Generale</u>

Des Surfaces by Gaston Darboux. In this chapter Darboux accomplishes more with the single second order linear hyperbolic equation than is available to us with a system of such equations, due to such causes as the non-commutativity of the ring of $n \times n$ matrices over the field of functions of two real variables. It is of course probable that some of the results obtained by Darboux which were not considered by us in this section, could be obtained for systems by matrix methods. The reader is referred to this very fine work by Darboux which has been the inspiration for much of this thesis.

SECTION IV

HYPERBOLIC EQUATIONS OF THIRD ORDER IN THREE INDEPENDENT VARIABLES

A. Let us now turn our attention to the linear equations of third order in three independent variables which have the form

$$(1) \quad \mathcal{L}(u) = u_{xyz} + au_{yz} + bu_{xz} + cu_{xy} + du_x + eu_y + fu_z + gu = 0 .$$

The coefficients a, b, c, d, e, f, and g are to be considered as functions of x, y, and z, continuously differentiable as many times as we may need. We wish to attack this problem in the manner of Laplace discussed previously, in the hope of reducing (1) to a system of three first order equations. Failing in this we will then attempt to cascade the equations, in the hope that after a finite number of iterations the chain will terminate with vanishing invariants, and that the resulting system can then be reduced as was originally desired.

To commence this operation, we must consider substitutions of three different types, namely

$$(2) \qquad u_1 = u_x + au ,$$

$$(3) \qquad u_1^2 = u_{1_y} + bu_1 ,$$

$$(4) \qquad u_{-1} = u_{yz} + bu_z + cu_y + du.$$

Let us start with the change of variables (2). Using the definition of the operator $\mathcal{L}$ which appears in (1), we readily verify that

$$(5)\qquad u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1 = \mathcal{L}(u) + k_1 u_y + l_1 u_z + m_1 u$$

where
$$k_1 = a_z + ac - e,$$
$$(6)\qquad l_1 = a_y + ab - f,$$
$$m_1 = a_{yz} + ca_y + ba_z + a_{yz} - g$$

are defined as the first three x-invariants. It is apparent from the symmetry of the equation that we could have just as well made the substitutions

$$(2')\qquad u_2 = u_y + bu \qquad \text{or}$$
$$(2'')\qquad u_3 = u_z + cu.$$

These lead to equations similar to (5) with the y-invariants (arising from (2')) being

$$h_2 = b_z + bc - d,$$
$$l_2 = b_x + ba - f,$$
$$m_2 = b_{xz} + cb_x + ab_z + be - g,$$

and the z-invariants (arising from (2'')) being

$$h_3 = c_y + cb - d,$$
$$k_3 = c_x + ca - e,$$
$$m_3 = c_{xy} + bc_x + ac_y + cf - g.$$

It is also apparent that whatever results we obtain from substitution (2)

would be symmetrically obtained from substitutions (2') and (2''), hence at this point we will limit the majority of the discussion to the effects of substitution (2).

Should we be rewarded by finding $k_1 = h_1 = m_1 = 0$, then (5) will become a second order hyperbolic equation in only two variables

$$u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1 = 0. \tag{7}$$

We may then apply the Laplace method to (7), as described briefly in Section I, or in more detail in the works of Darboux.[18]

Consider next the change of variables (3), which leads to the equation

$$u^2_{1_z} + au^2_1 = \mathcal{L}(u) + h_2u_x + k_1u_y + l_1u_z + m^2_1 u, \tag{8}$$

where now $m^2_1 = a_{yz} + (ab)_z + ca_y + abc - g$. As before, the symmetry of the equation indicates that the following changes of variables would lead to equations similar to (8):

$$\begin{aligned}
u^3_1 &= u_{1_z} + cu_1, \\
u^1_2 &= u_{2_x} + au_2, \\
u^3_2 &= u_{2_z} + cu_2, \\
u^1_3 &= u_{3_x} + au_3, \\
u^2_3 &= u_{3_y} + bu_3.
\end{aligned} \tag{3'}$$

These substitutions lead to the respective invariants

$$m_1^3 = a_{yz} + (ac)_y + ba_z + abc - g,$$

$$m_2^1 = b_{xz} + (ab)_z + cb_x + abc - g,$$

$$m_2^3 = b_{xz} + (bc)_x + ab_z + abc - g,$$

$$m_3^1 = c_{xy} + (ac)_y + bc_x + abc - g,$$

$$m_3^2 = c_{xy} + (bc)_x + ac_y + abc - g.$$

In the case of equation (8), if $k_1 = l_1 = h_2 = m_1^2 = 0$, we will have reduced our system immediately to three first order equations

$$\text{(9)} \qquad \begin{aligned} u_x + au &= u_1, \\ u_{1_y} + bu_1 &= u_1^2, \\ u^2_{1_z} + cu_1^2 &= 0, \end{aligned}$$

which can be solved in inverse order by quadratures.

Thirdly we consider the change of variables (4), which leads to the equation

$$\text{(10)} \qquad u_{-1_x} + au_{-1} = \mathcal{L}(u) + l_2 u_y + k_3 u_z + m_{-1}\, u.$$

where $m_{-1} = d_x + ad - g$ is the first <u>-x-invariant</u>. Once again symmetry considerations show that the change of variables

$$u_{-2} = u_{xz} + au_z + cu_x + eu,$$

and

$$u_{-3} = u_{xy} + au_y + bu_x + fu,$$

lead to the <u>-y-invariant</u> and <u>-z-invariant</u>

$$m_{-2} = e_y + eb - g,$$

$$m_{-3} = f_z + fc - g,$$

respectively.

If we are fortunate once more, such that $l_2 = k_3 = m_{-1} = 0$, we may solve (10) by quadratures, and then apply the Laplace method to the non-homogeneous equation (4).

B. We have in the discussion of paragraph A encountered some eighteen expressions which we have labeled "invariants". Organizing these in somewhat more orderly fashion, we have the following table:

$$h_2 = b_z + bc - d$$

$$h_3 = c_y + bc - d$$

$$k_1 = a_z + ac - e$$

$$k_3 = c_x + ac - e$$

$$l_1 = a_y + ab - f$$

$$l_2 = b_x + ab - f$$

$$m_1 = a_{yz} + ba_z + ca_y + ad - g$$

$$m_2 = b_{xz} + ab_z + cb_x + be - g$$

$$m_3 = c_{xy} + ac_y + bc_x + cf - g$$

$$m_{-1} = d_x + ad - g$$

$$m_{-2} = e_y + be - g$$

$$m_{-3} = f_x + cf - g$$

$$m_1^2 = a_{yz} + (ab)_z + ca_y + abc - g$$

$$m_1^3 = a_{yz} + (ac)_y + ba_z + abc - g$$

$$m_2^1 = b_{xz} + (ab)_z + cb_x + abc - g$$

$$m_2^3 = b_{xz} + (bc)_x + ab_z + abc - g$$

$$m_3^1 = c_{xy} + (ac)_y + bc_x + abc - g$$

$$m_3^2 = c_{xy} + (bc)_x + ac_y + abc - g.$$

We now wish to investigate the character of these eighteen "invariants", as was done in paragraph B of Section III for systems, in order to see if the term "invariant" is appropriate. That is, we are concerned with the change in these expressions when we make the various changes of variables

(11) $\qquad u' = \lambda u$, where $\lambda = \lambda(x,y,z)$ is at least twice continuously differentiable;

(12) $$x = \phi(x'),\quad y = \psi(y'),\quad z = \zeta(z');$$

(13) $\qquad x = y', \quad y = z', \quad z = x'.$

$$[illegible]$$

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Let us consider first the change of variables (11). Equation (1) is then transformed into a new equation of the same type

(14) $$\mathcal{L}'(u') = u'_{xyz} + a'u'_{yz} + b'u'_{xz} + c'u'_{xy} + d'u'_x + e'u'_y + f'u'_z + g'u' = 0,$$

but with new coefficients

(15) $$\begin{aligned} a' &= a + \frac{1}{\lambda}\frac{\partial \lambda}{\partial x} \\ b' &= b + \frac{1}{\lambda}\frac{\partial \lambda}{\partial y} \\ c' &= c + \frac{1}{\lambda}\frac{\partial \lambda}{\partial z} \\ d' &= d + \frac{c}{\lambda}\frac{\partial \lambda}{\partial y} + \frac{b}{\lambda}\frac{\partial \lambda}{\partial z} + \frac{1}{\lambda}\frac{\partial^2 \lambda}{\partial y \partial z} \\ e' &= e + \frac{c}{\lambda}\frac{\partial \lambda}{\partial x} + \frac{a}{\lambda}\frac{\partial \lambda}{\partial z} + \frac{1}{\lambda}\frac{\partial^2 \lambda}{\partial x \partial z} \\ f' &= f + \frac{b}{\lambda}\frac{\partial \lambda}{\partial x} + \frac{a}{\lambda}\frac{\partial \lambda}{\partial y} + \frac{1}{\lambda}\frac{\partial^2 \lambda}{\partial x \partial y} \\ g' &= g + \frac{1}{\lambda}\left(d\frac{\partial \lambda}{\partial x} + e\frac{\partial \lambda}{\partial y} + f\frac{\partial \lambda}{\partial z} + a\frac{\partial^2 \lambda}{\partial y \partial z} + b\frac{\partial^2 \lambda}{\partial x \partial z} + c\frac{\partial^2 \lambda}{\partial x \partial y} + \frac{\partial^3 \lambda}{\partial x \partial y \partial z}\right). \end{aligned}$$

Using these new coefficients, let us compute the three new x-invariant, k_1', l_1', and m_1'. We find

(16) $$\begin{aligned} k_1' &= a'_z + a'c' - e' = k_1, \\ l_1' &= a'_y + a'b' - f' = l_1, \\ m_1' &= a'_{yz} + b'a'_z + c'a'_y + a'd' - g' = m_1 + \frac{k_1}{\lambda}\frac{\partial \lambda}{\partial y} + \frac{l_1}{\lambda}\frac{\partial \lambda}{\partial z}. \end{aligned}$$

The new - x-invariant, m'_{-1}, becomes

(17) $$m'_{-1} = d'_x + a'd' - g' = m_{-1} + \frac{k_3}{\lambda}\frac{\partial \lambda}{\partial y} + \frac{l_2}{\lambda}\frac{\partial \lambda}{\partial z},$$

while the new m_1^2 invariant, $m_1^{2'}$, becomes

$$m_1^{2'} = a'_{yz} + (a'b')_z + c'a'_y + a'b'c' - g' =$$

(18)
$$= m_1^2 + \frac{h_2}{\lambda}\frac{\partial\lambda}{\partial x} + \frac{k_1}{\lambda}\frac{\partial\lambda}{\partial y} + \frac{l_1}{\lambda}\frac{\partial\lambda}{\partial z}.$$

From symmetry, similar results are obtained for the other h', k', l', and m' invariants. Thus we wee that the k, h, and l invariants, which appear as coefficients of the x, y, and z derivatives of u, respectively, are indeed true invariants under the change of variables (11). The m invariants, on the other hand, do not reproduce themselves exactly under this change of variables, but instead reproduce themselves plus a linear combination of the true invariants. Hence we shall in the future refer to these m invariants as <u>quasi-invariants</u>. (These will appear again in Section V.)

If we should make the change of variables (12) and compute the resulting invariants, we will find that

(19)
$$\begin{aligned} k_1' &= \Phi_{x'} Z_{z'}\; k_1, \\ l_1' &= \Phi_{x'} \Psi_{y'}\, l_1, \\ m_1' &= \Phi_{x'} \Psi_{y'} Z_{z'}\, m_1, \\ m_1^{2'} &= \Phi_{x'} \Psi_{y'} Z_{z'}\; m_1^2. \\ m_{-1}' &= \Phi_{x'} \Psi_{y'} Z_{z'}\; m_{-1}. \end{aligned}$$

Finally we note that (13) merely interchanges the x-invariants with the y-invariants, the y-invariants with the z-invariants, and the z-invariants with the x-invariants.

C. Let us now develop the cascade of equations in the manner of Laplace and Darboux. Consider first equation (5) in the event that at least one of

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the invariants is not zero. We wish to transform (5) into an equation for u_1 which is in the form (1), (See in this connection remark in footnote (5)) In order to do so, we proceed to integrate equation (2), obtaining

(20) $$u = e^{-\int a\,dx}\left[\int e^{\int a\,dx} u_1\,dx + X(y,z)\right],$$

where $X(y,z)$ is an arbitrary function of y and z only. Then differentiating (20) with respect to y and z yields

(21) $$u_y = e^{-\int a\,dx}\frac{\partial}{\partial y}\left[\int e^{\int a\,dx} u_1\,dx + X\right] - \frac{\partial(\int a\,dx)}{\partial y}\left[e^{-\int a\,dx}\left\{\int e^{\int a\,dx} u_1\,dx + X\right\}\right];$$

(22) $$u_z = e^{-\int a\,dx}\frac{\partial}{\partial z}\left[\int e^{\int a\,dx} u_1\,dx + X\right] - \frac{\partial}{\partial z}\left(\int a\,dx\right) e^{-\int a\,dx}\left[\int e^{\int a\,dx} u_1\,dx + X\right].$$

Substituting (20), (21), and (22) into (5) we obtain

(23) $$\begin{aligned} u_{1_{yz}} + b u_{1_z} + c u_{1_y} + d u_1 &= m_1 e^{-\int a\,dx}\left[\int e^{\int a\,dx} u_1\,dx + X\right] + \\ &+ k_1 e^{-\int a\,dx}\left[\frac{\partial}{\partial y}\left\{\int e^{\int a\,dx} u_1\,dx + X\right\} - \frac{\partial}{\partial y}\left(\int a\,dx\right)\left\{\int e^{\int a\,dx} u_1\,dx + X\right\}\right] + \\ &+ l_1 e^{-\int a\,dx}\left[\frac{\partial}{\partial z}\left\{\int e^{\int a\,dx} u_1\,dx + X\right\} - \frac{\partial}{\partial z}\left(\int a\,dx\right)\left\{\int e^{\int a\,dx} u_1\,dx + X\right\}\right]. \end{aligned}$$

Now (23) is an equation for u_1 alone, but it is an integral-differential equation, interlaced with arbitrary functions and their partial

derivatives, and hardly in a manageable state. Our aim is to reduce this to an equation for u_1 in the form of (1) and to this end we will differentiate (23) with respect to x. Before we do this, however, it will be necessary to make certain assumptions regarding the invariants and the coefficients. If these assumptions are made, the integral terms and the arbitrary functions will disappear when we differentiate. We will consider a number of different methods.

METHOD 1

The first assumptions in this method are

(a) $k_1 = l_1 = m_1 \neq 0$;

(b) a is a function of (x) only.

Under these hypotheses, (23) takes the form $u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1 =$

$$= m_1 e^{-\int adx}\left[\int e^{\int adx} u_1 dx + X + \frac{\partial}{\partial y}\left\{\int e^{\int adx} u_1 dx + X\right\} + \frac{\partial}{\partial z}\left\{\int e^{\int adx} u_1 dx + X\right\}\right].$$

Multiply both sides by $\dfrac{e^{\int adx}}{m_1}$, and differentiate with respect to x to get

$$\frac{a}{m_1} e^{\int adx}(u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1) -$$

(24)
$$-\frac{\partial \log m_1}{\partial x} \frac{e^{\int adx}}{m_1}(u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1) +$$

$$+ \frac{e^{\int adx}}{m_1}(u_{1_{xyz}} + bu_{1_{xz}} + cu_{1_{xy}} + du_{1_x} + c_x u_{1_y} + b_x u_{1_z} + d_x u_1) =$$

$$= e^{\int adx} u_1 + e^{\int adx} u_{1_y} + e^{\int adx} u_{1_z}.$$

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$$[illegible] \quad (1)$$

$$[illegible] \quad (2)$$

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Now we may multiply both sides of (24) by $m_1 e^{-\int a\,dx}$ and collect terms to obtain

$$u_{1_{xyz}} + \left(a - \frac{\partial \log m_1}{\partial x}\right) u_{1_{yz}} + bu_{1_{xz}} + cu_{1_{xy}} + du_{1_x} +$$

$$+ \left(c\left(a - \frac{\partial \log m_1}{\partial x}\right) + c_x - m_1\right) u_{1_y} + b\left(a - \frac{\partial \log m_1}{\partial x}\right) + b_x - m_1)u_{1_z} +$$

$$+ d\left(a - \frac{\partial \log m_1}{\partial x}\right) + d_x - m_1)\, u_1 = 0\,.$$

Designate the coefficients as follows:

$$a_1 = a - \frac{\partial \log m_1}{\partial x}\,, \qquad b_1 = b\,, \qquad c_1 = c\,, \qquad d_1 = d\,,$$

$$e_1 = c_1 a_1 + c_{1_x} - m_1\,, \qquad f_1 = b_1 a_1 + b_{1_x} - m_1\,,$$

$$g_1 = d_1 a_1 + d_{1_x} - m_1.$$

Then (25) is in the same form as (1):

$$(25) \qquad u_{1_{xyz}} + a_1 u_{1_{yz}} + b_1 u_{xz} + c_1 u_{1_{xy}} + d_1 u_{1_x} + e_1 u_{1y} + f_1 u_{1z} + g_1 u_1 = 0\,.$$

Our new x-invariants are, from the substitution $u_4 = u_{1_x} + a_1 u_1$,

$$k_4 = a_1 c_1 + a_{1_z} - e_1 = a_1 c_1 + a_{1_z} - a_1 c_1 - c_{1_x} + m_1 =$$

$$= a_{1_z} - c_{1_x} + m_1 =$$

$$= m_1 - \frac{\partial^2 \log m_1}{\partial z \partial x} - c_x.$$

Thus

$$k_4 = k_1 - \left(c_x + \frac{\partial^2 \log m_1}{\partial z \partial x}\right).$$

Also

$$l_4 = a_1b_1 + a_{1_y} - f_1 = a_1b_1 - \frac{\partial^2 \log m_1}{\partial y \partial x} - a_1b_1 + m_1 - b_{1_x} =$$

$$= l_1 - (b_x + \frac{\partial^2 \log m_1}{\partial z \partial x}),$$

$$m_4 = a_1d_1 + b_1a_{1_z} + c_1a_{1_y} + a_{1yz} - g_1 =$$

$$= a_1d_1 - a_1d_1 - d_{1_x} + m_1 - b_1 \frac{\partial^2 \log m_1}{\partial z \partial x} - c_1 \frac{\partial^2 \log m_1}{\partial y \partial x} -$$

$$- \frac{\partial^3 \log m_1}{\partial x \partial y \partial z} =$$

$$= m_1 - (d_x + b \frac{\partial^2 \log m_1}{\partial z \partial x} + c \frac{\partial^2 \log m_1}{\partial y \partial x} + \frac{\partial^3 \log m_1}{\partial x \partial y \partial z} .$$

If now the new x-invariants are all zero, we may proceed to reduce the equation as described in paragraph A and solve. If, however, at least one of the invariants is not zero, we are in a dilemma, for the hypothesis with which we transformed (5) into (25) are no longer valid in the general case. We can no longer continue our chain. To enable us to do so, we must change our original hypothesis to the following:

(a) $k_1 = l_1 = m_1 \neq 0$,

(b) All coefficients are functionsof x only,

(c) $b \equiv c \equiv d$.

Under these hypotheses, our invariants are also functions of x only. We

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may transform (5) into (25) as before, and the coefficients will have the same form. The x-invariants will be different, however, and in fact will be

$$
\begin{aligned}
k_4 &= k_1 - c_x = k_1 - d_x, \\
l_4 &= l_1 - b_x = l_1 - d_x, \qquad (26) \\
m_4 &= m_1 - d_x.
\end{aligned}
$$

Hence $k_4 = l_4 = m_4$, our hypothesis are again satisfied, and the chain may be continued.

Observe that, as our chain progresses, our x-invariants will always have the form

$$k_{1+3j} = l_{1+3j} = m_{1+3j} = m_1 - jd_x, \quad j = 0,1,2\ldots .$$

Thus, if our invariants are not originally all zero, they can only become zero if they are an integer multiple of the partial derivative of d with respect to x.

Note also, that to have a chain of y-invariants, or z-invariants, our hypotheses (a), (b), and (c) must be modified accordingly.

METHOD 2

In this method we assume first that

$$\text{(a)} \quad k_1 \equiv l_1 \equiv 0.$$

Under this hypothesis, equation (23) takes the form

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$$(27) \qquad u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1 = m_1 e^{\int -adx} \left[\int e^{\int adx} u_1 dx + X \right].$$

Multiply both sides by $\dfrac{e^{\int adx}}{m_1}$ and differentiate with respect to x to get

$$\frac{e^{\int adx}}{m_1} \left(a - \frac{\partial \log m_1}{\partial x} \right) (u_{1_{yz}} + bu_{1_z} + cu_{1_y} + du_1) +$$

$$+ \frac{e^{\int adx}}{m_1} (u_{1_{xyz}} + bu_{1_{xz}} + cu_{1_{xy}} + du_{1_x} + c_x u_{1_y} + b_x u_{1_z} + d_x u_1) =$$

$$= e^{\int adx} u_1 .$$

Then multiplying by $m_1 e^{-\int adx}$ and collecting terms, we obtain

$$u_{1_{xyz}} + \left(a - \frac{\partial \log m_1}{\partial x}\right) u_{1_{yz}} + bu_{1_{xz}} + cu_{1_{xy}} + du_{1_x} +$$

$$(28) \qquad + \left[c \left(a - \frac{\partial \log m_1}{\partial x}\right) + c_x\right] u_{1_y} + \left[b \left(a - \frac{\partial \log m_1}{\partial x}\right) + b_x\right] u_{1_z} +$$

$$+ \left[d \left(a - \frac{\partial \log m_1}{\partial x}\right) + d_x - m_1\right] u_1 = 0 .$$

As before, designate the coefficients of (28) as follows:

$$a_1 = a - \frac{\partial \log m_1}{\partial x}, \qquad b_1 = b, \qquad c_1 = c, \quad d_1 = d,$$

$$e_1 = c_1 a_1 + c_{1_x}, \qquad f_1 = b_1 a_1 + b_{1_x}, \quad g_1 = d_1 a_1 + d_{1x} - m_1 .$$

Then (28) is in the same form as (1), and is identical with (25).

Now, however, the x-invariants resulting from the substitution

$u_4 = u_{1_x} + a_1 u_1$ are

$$k_4 = a_1 c_1 + a_{1_z} - e_1 = a_1 c_1 - a_1 c_1 + a_{1_z} - c_{1_x}$$

$$= a_z - c_x - \frac{\partial^2 \log m_1}{\partial z \partial x},$$

$$l_4 = a_1 b_1 + a_{1_y} - f_1 = a_y - b_x - \frac{\partial^2 \log m_1}{\partial y \partial x},$$

$$m_4 = a_1 d_1 + b_1 a_{1_z} + c_1 a_{1_y} + a_{1_{yz}} - g_1 =$$

$$= m_1 - d_x + ba + ca_y + a_{yz} - b\frac{\partial^2 \log m_1}{\partial z \partial x} -$$

$$- c\frac{\partial^2 \log m_1}{\partial y \partial x} - \frac{\partial^3 \log m_1}{\partial z \partial y \partial x}.$$

If $k_4 \equiv l_4 \equiv m_4 \equiv 0$, we may reduce as before, and solve. If, however, at least one of these invariants is not zero, we are again faced with the dilemma that our original hypothesis is not satisfied in the most general case, and the chain cannot be continued. Thus, we must again modify the hypothesis in order to iterate and continue the chain. Our hypotheses become

(a) $k_1 \equiv l_1 \equiv 0$;

(b) $a = a(x)$, $b = b(y,z)$, $c = c(y,z)$,

$d = d(x)$, $e = e(x,y,z) = ac$, $f = f(x,y,z) = ab$,

$g = g(x)$.

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Under these hypotheses, $m_1 = ad - g$, is a function of x only, as is $\dfrac{\partial \log m_1}{\partial x}$. Now when we transform (1) into (25), the coefficients will be

$$a_1 = a - \frac{\partial \log m_1}{\partial x}, \quad b_1 = b, \quad c_1 = c, \quad d_1 = d,$$

$$e_1 = c_1 a_1, \quad f_1 = b_1 a_1, \quad g_1 = d_1 a_1 + d_{1_x} - m_1.$$

Thus we will find that

$$k_4 \equiv 0, \quad l_4 \equiv 0, \quad m_4 = m_1 - d_x,$$

hence, the hypotheses (a) and (b) are both satisfied once more, and the chain may be continued.

Observe that, as before, as the chain progresses, the invariants must have the form

$$k_{1+3i} \equiv l_{1+3i} \equiv 0, \quad m_{1+3i} = m_1 - id_x, \quad i = 0,1,2,\ldots .$$

Thus if $m_1 \neq 0$, the invariants for the $j^{\underline{th}}$ iteration will vanish only if m_1 is an integer multiple of d_x, i.e., $m_1 = j\, d_x$, for some $j = 0,1,2\ldots$ We also remark again, that to have a chain of y-invariants or z-invariants, hypotheses (a) and (b) must be modified accordingly.

D. Now we wish to consider methods for cascading the equations when we employ the substitution (4), in the event that at least one of the x-invariants is not identically zero. For this chain we wish to solve (10) for u in terms of u_{-1}, and, after the prescribed differentiations, substitute into (4) to obtain an equation of the form (25) in u_{-1}. In order to accomplish this we are again forced to place certain conditions

on the coefficients and the related invariants. Thus we introduce

Method 3: For this we initially assume that

$$\text{(a)} \qquad l_2 \equiv k_3 \equiv 0 .$$

Solving (10) for u, we obtain

$$u = \frac{1}{m_{-1}} (u_{-1_x} + au_{-1})$$

and hence

$$u_y = \frac{1}{m_{-1}} (u_{-1_{xy}} + au_{-1_y} + a_y u_{-1}) - \frac{1}{m_{-1}^2} \frac{\partial m_{-1}}{\partial y} (u_{-1_x} + au_{-1}),$$

$$u_z = \frac{1}{m_{-1}} (u_{-1_{xz}} + au_{-1_z} + a_z u_{-1}) - \frac{1}{m_{-1}^2} \frac{\partial m_{-1}}{\partial z} (u_{-1_x} + au_{-1}),$$

$$u_{yz} = \frac{1}{m_{-1}} (u_{-1_{xyz}} + au_{-1_y} + a_y u_{-1_z} + a_z u_{-1_y} + a_{yz} u_{-1}) -$$

$$- \frac{1}{m_{-1}^2} \frac{\partial m_{-1}}{\partial y} (u_{-1_{xz}} + au_{-1z} + a_z u_{-1}) - \frac{1}{m_{-1}^2} \frac{\partial m_{-1}}{\partial z} (u_{-1_{xy}} + au_{-1_y} + a_y u_{-1})$$

$$- \frac{1}{m_{-1}^2} \frac{\partial^2 m_{-1}}{\partial y \partial z} (u_{-1_x} + au_{-1}) + \frac{2}{m_{-1}^3} \frac{\partial m_{-1}}{\partial y} \frac{\partial m_{-1}}{\partial z} (u_{-1_x} + au_{-1}).$$

Substituting these expressions into (4) and collecting terms we obtain

$$(29) \qquad u_{-1_{xyz}} + a_{-1} u_{-1yz} + b_{-1} u_{-1_{xz}} + c_{-1} u_{-1_{xy}} + d_{-1} u_{-1_x} + e_{-1} u_{-1_y} +$$

$$+ f_{-1} u_{-1_z} + g_{-1} u_{-1} = 0,$$

where $a_{-1} = a$, $b_{-1} = b - \frac{\partial \log m_{-1}}{\partial y}$, $c_{-1} = c - \frac{\partial \log m_{-1}}{\partial z}$,

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$$d_{-1} = d - c\,\frac{\partial \log m_{-1}}{\partial y} - b\,\frac{\partial \log m_{-1}}{\partial z} - \frac{\partial^2 \log m_{-1}}{\partial y \partial z} +$$

$$+ \frac{\partial \log m_{-1}}{\partial y}\,\frac{\partial \log m_{-1}}{\partial z} ,$$

$e_{-1} = a_{-1_z} + a_{-1}c_{-1}$, $f_{-1} = a_{-1_y} + a_{-1}b_{-1}$, and

$g_{-1} = a_{-1_{yz}} + b_{-1}a_{-1_z} + c_{-1}a_{-1_y} + a_{-1}d_{-1} - m_{-1}$,

and (29) is of the desired form. Computing the new $-x$-invariants, we obtain

$$l_5 = b_x - a_y - \frac{\partial^2 \log m_{-1}}{\partial x \partial y} ,$$

$$k_6 = c_x - a_z - \frac{\partial^2 \log m_{-1}}{\partial x \partial z} ;$$

and

$$m_{-4} = m_{-1} + d_x - \frac{\partial}{\partial x}\left(c\,\frac{\partial \log m_{-1}}{\partial y} + b\,\frac{\partial \log m_{-1}}{\partial z} + \frac{\partial^2 \log m_{-1}}{\partial y \partial z} -\right.$$

$$\left. - \frac{\partial \log m_{-1}}{\partial y}\,\frac{\partial \log m_{-1}}{\partial z}\right) - a_y\left(c - \frac{\partial \log m_{-1}}{\partial z}\right) -$$

$$- a_z\left(b - \frac{\partial \log m_{-1}}{\partial y}\right) + a_{yz} .$$

As in paragraph C, we will be in a dilemma if the new $-x$-invariants are not all zero, for then our initial hypothesis (a) will not be satisfied in general, by (29), and our chain cannot continue. To escape this trap, we once more impose additional restrictions on our coefficients. Our revised hypotheses will be

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$$\text{(a)} \quad l_2 \equiv k_3 \equiv 0 \; ;$$

$$\text{(b)} \quad a = a(x), \quad b = b(y,z), \quad c = c(y,z) \; ,$$

$$d = d(x), \quad e = e(x,y,z) = a(x)c(y,z) \; ,$$

$$f = f(x,y,z) = a(x)b(y,z) \; , \quad g = g(x) \; .$$

Under hypotheses (a) and (b), m_{-1} becomes a function of x only, and hence the coefficients of (29) become

$$a_{-1} = a, \quad b_{-1} = b, \quad c_{-1} = c, \quad d_{-1} = d, \quad e_{-1} = e, \quad f_{-1} = f$$

$$g_{-1} = ad - m_{-1} \; .$$

With these coefficients, the -x-invariants for (29) are

$$l_5 \equiv k_6 \equiv 0, \quad m_{-4} = m_{-1} + d_x \; ,$$

which means our hypotheses (a) and (b) are again satisfied, and the chain may be continued. As observed in methods 1, and 2, the invariant $m_{-1} - 3j$ can vanish only if m_{-1} is an integer multiple of d_x, that is, $m_{-1} = -jd_x$. We may add at this point, that if m is a positive multiple of d_x, we should use substitution (2), while if it is a negative multiple of d_x, substitution (4) would be the more advantageous. One more it is worthwhile to note that if we desire a chain of - y - invariants, or -z- invariants, hypotheses (a) and (b) must be modified accordingly.

E. Finally we consider a method for cascading the equations when we employ the substitution (3), in the event that at least one of the invariants in (8) is not zero. Although this substitution appears the most natural

in that our equation (1) is immediately broken down into three first order equations, we will find that the most stringent conditions on the coefficients are required in this case in order to generate the chain.

Method 4: Our initial assumption in this method is that (a) $h_2 \equiv k_1 \equiv l_1 \equiv 0$. We must consider the system of equations

$$u_x + au = u_1 ,$$
$$u_{1_y} + bu_1 = u_1^2 ,$$
$$u_{1_z}^2 + cu_1^2 = m_1^2 u .$$

If we solve the second equation for u_1, substitute this value into the first equation and solve for u, and then substitute the resulting expression into the third equation, we obtain

$$u_{1_z}^2 + cu_1^2 = m_1^2 e^{-\int adx}\left[\int e^{\int adx - bdy}\left\{\int e^{\int bdy} u_1^2 dy + Y(x,z)\right\} dx + X(y,z)\right],$$

where Y and X are arbitrary functions of their respective arguments. We then proceed as before:

$$\frac{e^{\int adx}}{m_1^2}\left[u_{1_z}^2 + cu_1^2\right] = \int e^{\int adx - bdy}\left\{\int e^{\int bdy} u_1^2 dy + Y\right\} dx + X.$$

Taking the partial derivative of both sides with respect to x,

$$(30) \quad \frac{e^{\int adx}}{m_1^2}\left[\left(a - \frac{\partial \log m_1^2}{\partial x}\right)(u_{1_z}^2 + cu_1^2) + u_{1_{xz}}^2 + cu_{1_x}^2 + c_x u_1^2\right] =$$
$$= e^{\int adx - bdy}\left\{\int e^{\int bdy} u_1^2, dy + Y\right\}.$$

We then multiply both sides of (30) by $e^{\int bdy - adx}$ and take the partial derivative with respect to y, to obtain

$$\frac{e^{\int bdy}}{m_1^2}\Big[(b - \frac{\partial \log m_1^2}{\partial y})\Big\{(a - \frac{\partial \log m_1^2}{\partial x})(u^2_{1_z} + cu_1^2) + u^2_{1_{xz}} + cu^2_{1_x} + c_x u_1^2\Big\} + \tag{31}$$
$$+ (a_y - \frac{\partial^2 \log m_1^2}{\partial x \partial y})(u^2_{1_z} + cu_1^2) + (a - \frac{\partial \log m_1^2}{\partial x})(u^2_{1_{yz}} + cu^2_{1_y} + c_y u_1^2) +$$
$$u^2_{1_{xyz}} + cu^2_{1_{xy}} + c_y u^2_{1_x} + c_x u^2_{1_y} + c_{xy} u_1^2\Big] = e^{\int bdy} u_1^2 .$$

Finally, multiplying (31) by $m_1^2 e^{\int -bdy}$, and collecting terms, we obtain an equation for u_1^2, which is of the same form as (1), but whose coefficients are $a_1^2 = a - \frac{\partial \log m_1^1}{\partial x}$; $b_1^2 = b - \frac{\partial \log m_1^1}{\partial y}$; $c_1^2 = c$;

$$d_1^2 = a_1^2 c_1^2 + c^2_{1_y}; \quad e_1^2 = a_1^2 c_1^2 + c^2_{1_x} \; ; \; f_1^2 = a_1^2 b_1^2 + a^2_{1_y} \; ;$$

$$g_1^2 = a_1^2 b_1^2 c_1^2 + a_1^2 b_1^2 c^2_{1_x} + (a_1^2 c_1^2)_y + c^2_{1_{xy}} - m_1^2 .$$

Computing the new invariants, we find

$$h_5 = b^2_{1_z} + b_1^2 c_1^2 - d_1^2 = b^2_{1_z} - c^2_{1_y} ,$$

$$k_4 = a^2_{1_z} + a_1^2 c_1^2 - e_1^2 = a^2_{1_z} - c^2_{1_x} ,$$

$$l_4 = a_1^2 + a_1^2 b_1^2 - f_1^2 \equiv 0 ,$$

$$m_4^2 = a^2_{1_{yz}} + (a_1^2 b_1^2)_z + c_1^2 a^2_{1_y} + a_1^2 b_1^2 c_1^2 - g_1^2 =$$

$$= m_1^2 + a^2_{1_{yz}} + (a_1^2 b_1^2)_z + a^2_{1_y} c_1^2 + c^2_{1_{xy}} .$$

We are faced with the same difficulty as before. In general, the invariants h_5 and k_4 will not vanish, hence it will be impossible to

continue the chain in this manner. We will be able to continue the chain if we revise our hypothesis to be (a) $h_2 \equiv k_1 \equiv l_1 \equiv 0$,

(b) All coefficients are functions of y only, except c, and that c is a constant.

With these hypotheses our new coefficients become

$$a_1^2 = a; \quad b_1^2 \quad b - \frac{\partial \log m_1^2}{\partial y}; \quad c_1^2 = c\ ; \quad d_1^2 = b_1^2 c_1^2\ ;$$

$$e_1^2 = a_1^2 c_1^2; \quad f_1^2 = a_1^2 b_1^2 + a_{1y}^2, \quad g_1^2 = a_1^2 b_1^2 c_1^2 - m_1^2\ ,$$ and the

new invariants become

$$h_5 \equiv k_4 \equiv l_4 \equiv 0\ ,$$

$$m_4^2 = m_1^2 + ca_y\ .$$

Thus the hypotheses (a) and (b) are satisfied once more, and the chain may be continued. We observe that in this method the chain will terminate with all invariants zero if and only if m_1^2 is a negative integer multiple of ca_y.

These methods described in paragraphs C, D, and E are not, of course, the only methods by which chains of equations could be generated. We could do a method similar to method 1, say, with $k_1 \equiv 0$ and $l_1 = m_1$; or we could do a method similar to method 2 with k_1 the non-zero invariant, or l_1 the non-zero invariant. There are many variations on the theme presented here, but it is the theme itself which is the important thing. In any such method certain restrictions of the coefficients are necessary to iterate once, and other restrictions are necessary to iterate more than

once. The basic idea is to find an expression for u in terms of the new dependent variable, which may or may not be an integral expression. We then differentiate this expression as required by the particular equation, i.e. the equation involving the invariant coefficients. Substituting the resulting expressions into this equation, we reduce it to an equation in the new dependent variable alone; if necessary, we separate these expressions from the invariant coefficients and differentiate again sufficiently to remove any integral signs which may appear. If we can accomplish this, we can in general iterate the equation.

F. We consider now the most general form of (1) which can be reduced by these methods described in paragraphs C, D, and E. The question is, what must the coefficients of (1) be, in order that (1) may be reduced to a second order equation by these methods, when the x-invariants or the -x-invariants are not all originally zero? We observe first, in methods 1, 2, and 3, that in order for the m-invariants to vanish after n-1 iterations we must have

$$g - ad = nd_x, \quad \text{where} \quad n = \pm 1, \pm 2, \ldots$$

or

$$d_x + \frac{a}{n} d = \frac{g}{n} . \tag{32}$$

Equation (32) is integrable, and integration gives us

$$d = e^{-\int dx} \int e^{\int dx} \frac{g}{n} dx \quad \text{plus an arbitrary constant} \tag{33}$$

which we choose to be zero.

Using (33) and the hypotheses of method 1, we see that our original equation must be

$$u_{xyz} + a(x)u_{yz} + e^{-\int\frac{a}{n}dx}\int e^{\int\frac{a}{n}dx}\frac{g}{n}\,dx\left(u_{xz}+u_{xy}+u_x\right) + \tag{34}$$

$$+ g(x)(u_y + u_z + u) = 0 \quad , \quad n = 1,2, \ldots.$$

for method 1 to be applicable. Using (33) and the hypotheses of methods 2 and 3, we see that our original equation must be

$$u_{xyz} + a(x)u_{yz} + b(y,z)u_{xz} + c(y,z)u_{xy} + \left(e^{-\int\frac{a}{n}dx}\int e^{\int\frac{a}{n}dx}\frac{g}{n}\,dx\right)u_x + \tag{35}$$

$$+ a(x)c(y,z)u_y + a(x)b(y,z)u_z + g(x)u = 0 \;, \quad n = 1,2,\ldots$$

for methods 2 or 3 to be applicable.

Finally we consider the form of the coefficients when method 4 will be applicable. In this case we observe that in order for the m invariant to vanish after n-1 iterations, we must have

$$g - abc = nca_y \;, \quad n = 1,2,\ldots$$

or

$$a_y + \frac{b}{n}a = \frac{g}{nc}. \tag{36}$$

Equation (36) may be integrated to give

$$a = e^{-\int\frac{b\,dy}{n}}\int e^{\int\frac{b\,dy}{n}}\frac{g}{c}\frac{dy}{n}. \tag{37}$$

Using (37) and the hypotheses of method 4, we see that our original equation must be

$$u_{xyz} + \left(e^{-\int\frac{b\,dy}{n}}\int e^{\int\frac{b\,dy}{n}}\frac{g}{c}\frac{dy}{n}\right)u_{yz} + b(y)u_{xz} + cu_{xy} +$$

$$+ cb(y)u_x + \left(ce^{-\int\frac{b\,dy}{n}}\int e^{\int\frac{b\,dy}{n}}\frac{g}{c\,n}\,dy\right)u_y + \tag{38}$$

$$+ \left(\frac{g}{c\ n} + \frac{b(n-1)}{n} e^{-\int \frac{b}{n} dy} \int e^{\int \frac{b}{n} dy} \frac{g}{c\ n}\, dy \right) u_z + g(y) u = 0,$$

for method 4 to be applicable.

SECTION V

HYPERBOLIC EQUATIONS OF N^{TH} ORDER IN N INDEPENDENT VARIABLES

A. Having considered the extension of the Laplace cascade method to the equation of third order in three variables of the mixed derivatives type, we will now turn our attention to a generalization of these results. That is, we consider an equation of n^{th} order in n independent variables, in which only the mixed derivatives appear, and we shall attempt to show how these methods can be applied to such an equation.

Before proceeding to the n^{th} order equation we will first introduce an operator notation which will simplify somewhat the invariant expressions and relations, which tend to become awkward as we increase n. In order to do so, we mention here a rule which has been devised by J. Le Roux[19] and shall hereafter be referred to as Le Roux's rule:

"For any differential operator D which is a polynomial in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ the following relation holds:

$$D(uv) = uD(v) + \frac{\partial u}{\partial x} D'_x(v) + \frac{\partial u}{\partial y} D'_y(v) + \frac{1}{2!}\left[\frac{\partial^2 u}{\partial x^2} D''_{xx}(v) + 2\frac{\partial^2 u}{\partial x \partial y} D''_{xy}(v) + \frac{\partial^2 u}{\partial y^2} D''_{yy}(v)\right] +$$

$$+ \frac{1}{3!}\left[\frac{\partial^3 u}{\partial x^3} D'''_{xxx}(v) + 3\frac{\partial^3 u}{\partial x^2 \partial y} D'''_{xxy}(v) + 3\frac{\partial^3 u}{\partial x \partial y^2} D'''_{xyy}(v) + \frac{\partial^3 u}{\partial y^3} D'''_{yyy}(v)\right] + \cdots$$

where $D'_x = \dfrac{\partial D}{\partial\left(\frac{\partial}{\partial x}\right)}$, $D'_y = \dfrac{\partial D}{\partial\left(\frac{\partial}{\partial y}\right)}$, etc."

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This rule is an extension of Taylor's formula, and is, of course, completely analogous to Leibnitz' Rule for the n^{th} partial derivative of a product u v. It can be similarly extended to products of more than two functions. As an example of this rule, consider the operator $D(u) = u_{xy} + au_x + bu_y + cu = 0$. Here $D = \frac{\partial}{\partial x}\frac{\partial}{\partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c$, hence

$$D'_x = \frac{\partial}{\partial y} + a \;, \quad D'_y = \frac{\partial}{\partial x} + b \;, \quad D''_{xy} = I, \text{ the identity operator.}$$

For this operator D, it is a simple matter to verify the identities

$$(1) \qquad D'_y D'_x(u) = D(u) + hu,$$

$$(2) \qquad D'_x D'_y(u) = D(u) + ku,$$

where $h = a_x + ab - c$, $k = b_y + ab - c$ are the two Darboux invariants. Then denoting $u_1 = D'_x(u)$ and observing that $D(u) = 0$, we have from (1)

$$(3) \qquad D'_y(u_1) = hu .$$

Operating on (3) with D'_x, we obtain

$$(4) \qquad D'_x D'_y(u_1) = D'_x(hu).$$

Using the identity (2) to evaluate the left-hand side of (4) and Le Roux's rule to evaluate the right-hand side, we find

$$\begin{aligned} D(u_1) + ku_1 &= h\, D'_x(u) + \frac{\partial h}{\partial x} D''_{xx}(u) + \frac{\partial h}{\partial y} D''_{xy}(u) \\ &= h\, u_1 + \frac{\partial h}{\partial y} u \\ &= h\, u_1 + \frac{\partial h}{\partial y} \cdot \frac{h}{h} \cdot u \\ &= h\, u_1 + \frac{\partial \log h}{\partial y} D'_y(u_1), \end{aligned}$$

or

$$D(u_1) - \frac{\partial \log h}{\partial y} D'_y(u_1) + (k-h)u_1 = 0 .$$

This may be written in the form $D_1(u_1) = 0$

where $D_1 = \frac{\partial}{\partial x}\frac{\partial}{\partial y} + a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1$,

$$a_1 = a - \frac{\partial \log h}{\partial y},$$

$$b_1 = b,$$

$$c_1 = c - b\frac{\partial \log h}{\partial y} + b_y - a_x,$$

and these coefficients agree exactly with those obtained by Darboux[20] for the first equation of the cascade method.

Our problem now is to expand these procedures to the n^{th} order equation in n independent variables, of the mixed derivative type. Can we in this manner predict the form of the "invariants", the corresponding identities, and methods for cascading the equations?

B. Let us, for convenience sake, change the notation slightly. Instead of indicating the independent variables by $x, y, z, \ldots$, we will find it more advantageous to label them $x_1, x_2, \ldots x_n$. Then let the n^{th} order operator with which we will be concerned be defined by

$$(5) \qquad D(u) = \sum a_{i_1 i_2 \cdots i_n} u_{i_1 i_2 \cdots i_n},\ i_j = 0, 1, 2, \ldots, n$$

where we always require that $i_j \leqq i_{j+1}$ and that $i_j = i_{j+1}$ if and only if $i_{j+1} = 0$. The sum is to be taken over all acceptable combinations of the i_j where $i_j = 0, \ldots, n$. In (5) we write $u_j = \frac{\partial u}{\partial x_j}$, $j = 0, 1, \ldots n$ and define $\frac{\partial u}{\partial x_0} \equiv u$. Finally we require that in all of these operators we will set

$a_{12\cdots n} = 1$. To illustrate, let $n = 2$. Then

$$D(u) = \frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{01} \frac{\partial u}{\partial x_1} + a_{02} \frac{\partial u}{\partial x_2} + a_{00}u.$$

For $n = 3$ we have

$$D(u) = \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} + a_{023} \frac{\partial^2 u}{\partial x_2 \partial x_3} + a_{013} \frac{\partial^2 u}{\partial x_1 \partial x_3} + a_{012} \frac{\partial^2 u}{\partial x_1 \partial x_2} +$$

$$+ a_{001} \frac{\partial u}{\partial x_1} + a_{002} \frac{\partial u}{\partial x_2} + a_{003} \frac{\partial u}{\partial x_3} + a_{000}u,$$

and the brevity of (5) becomes more apparent with increasing n. Having seen these examples in explanation, there will be no loss of clarity if we drop the zero subscripts of the a's wherever they may appear. Thus for $n = 2$, we write $D(u) = u_{12} + a_2u_2 + a_1u_1 + au$.

In order to point the way, let us briefly summarize the 2^{nd} order results of Darboux and our 3^{rd} order results in the light of this notation. For the second order equation we have the following operators:

$$D = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} + a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a,$$

$$D_1' = \frac{\partial}{\partial x_2} + a_1,$$

$$D_2' = \frac{\partial}{\partial x_1} + a_2,$$

$$D_{12}'' = I.$$

The two Darboux invariants are given by

$$h = \frac{\partial}{\partial x_1} a_1 + a_2 a_1 - a = D_2'(a_1) - a,$$

$$k = \frac{\partial}{\partial x_2} a_2 + a_1 a_2 - a = D_1'(a_2) - a,$$

while the identities (1) and (2) become, respectively

$$D_2' D_1'(u) = D(u) + hu = D(u) + \left\{ D_2'(a_1) - a \right\} u,$$

$$D_1' D_2'(u) = D(u) + ku = D(u) + \left\{ D_1'(a_2) - a \right\} u.$$

For the 3^{rd} order equation, the operators are as follows:

$$D = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} + a_{12}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} + a_{13}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} + a_{23}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} +$$

$$+ a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} + a,$$

$$D_1' = \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} + a_{12}\frac{\partial}{\partial x_2} + a_{13}\frac{\partial}{\partial x_3} + a_1,$$

$$D_2' = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3} + a_{12}\frac{\partial}{\partial x_1} + a_{23}\frac{\partial}{\partial x_3} + a_2,$$

$$D_3' = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} + a_{13}\frac{\partial}{\partial x_1} + a_{23}\frac{\partial}{\partial x_2} + a_3,$$

$$D_{12}'' = \frac{\partial}{\partial x_3} + a_{12},$$

$$D_{13}'' = \frac{\partial}{\partial x_2} + a_{13},$$

$$D_{23}'' = \frac{\partial}{\partial x_1} + a_{23},$$

and associated with this equation are the invariants

$$h_2 = D''_{12}(a_{13}) - a_1 \qquad\qquad m_1^2 = D''_{12}D''_{13}(a_{23}) - a$$

$$h_3 = D''_{13}(a_{12}) - a_1 \qquad\qquad m_1^3 = D''_{13}D''_{12}(a_{23}) - a$$

$$k_1 = D''_{12}(a_{23}) - a_2 \qquad\qquad m_2^1 = D''_{12}D''_{23}(a_{13}) - a$$

$$k_3 = D''_{23}(a_{12}) - a_2 \qquad\qquad m_2^3 = D''_{23}D''_{12}(a_{13}) - a$$

$$l_1 = D''_{13}(a_{23}) - a_3 \qquad\qquad m_3^1 = D''_{13}D''_{23}(a_{12}) - a$$

$$l_2 = D''_{23}(a_{13}) - a_3 \qquad\qquad m_3^2 = D''_{23}D''_{13}(a_{12}) - a$$

$$m_1 = D'_1(a_{23}) - a \qquad\qquad m_{-1} = D''_{23}(a_1) - a$$

$$m_2 = D'_2(a_{13}) - a \qquad\qquad m_{-2} = D''_{13}(a_2) - a$$

$$m_3 = D'_3(a_{12}) - a \qquad\qquad m_{-3} = D''_{12}(a_3) - a\,.$$

A typical identity involving these invariants is

$$D'_1D''_{23}(u) = D(u) + (D''_{12}(a_{23}) - a_2)u_2 + (D''_{13}(a_{23}) - a_3)u_3 + (D'_1(a_{23}) - a)u\,.$$

C. Now let us generalize these results to the n^{th} order linear differential equation of the form (5). In order to demonstrate the magnitude of the problem, let us first compute $N(n)$, the number of different identities associated with the n^{th} order equation, $D(u) = 0$.

An identity for $D(u)$ is a combination of differentiated operators D', D'',... such that successive application of these primed operators to a function $u = u(\bar{x})$, where $\bar{x} = (x_1, x_2, \ldots, x_n)$, will yield $D(u)$ plus a linear combination of the mixed derivatives of u of order less than n. We shall denote such a linear combination by R, for remainder, and define the coefficients of R to be the invariant coefficients (or simply <u>invariants</u>)

associated with the operator D. We shall show presently that the order of R must be less than n-1, although this is not evident a priori. To be a systematic we proceed as follows:

Select arbitrarily a primed operator having order n-1, i.e. D_i' for some $i=1,2,\ldots,n$. There are, of course, n choices for this operator. To obtain an identity as defined above, there is only one choice of operator $D^{(p)}$ such that $D_i' D^{(p)}(u) = D(u) + R$. This operator must be $D^{(n-1)}_{1\,2\ldots i-1\; i+1\ldots n}$, since this contains the differentiation $\frac{\partial}{\partial x_i}$ which is missing from D_i'. Therefore we have exactly n identities of the form

$$D_i' D^{(n-1)}_{1\ldots i-1\; i+1\ldots n}(u) = D(u) + R, \qquad i=1,2,\ldots,n.$$

Next select arbitrarily a primed operator of order n-2, D''_{ij} for some $i=1,2,\ldots n$ and some $j=1,2,\ldots,n$, $j\neq i$. There are n choices for i, and n-1 choices for j, but since $D''_{ij} = D''_{ji}$, we total only $\frac{n(n-1)}{2} = \binom{n}{2}$ such operators, where $\binom{n}{i}$ denotes the appropriate binomial coefficient. If we wish to obtain an identity when we apply such an operator, we must apply it to one of threethings:

1) $D^{(n-2)}_{1\,2\ldots i-1\; i+1\ldots j-1\; j+1\ldots n}(u)$.

2) $D^{(n-1)}_{1\,2\ldots i-1\; i+1\ldots n} D^{(n-1)}_{1\,2\ldots j-1\; j+1\ldots n}(u)$.

3) $D^{(n-1)}_{1\,2\ldots j-1\; j+1\ldots n} D^{(n-1)}_{1\,2\ldots i-1\; i+1\ldots n}(u)$.

Therefore we have $3\binom{n}{2}$ identities of the form

$$D''_{ij}\ L(u) = D(u) + R$$

where $L(u)$ is one of the three expression shown above.

If we continue to count the identities in this manner, we find that

$$(6) \qquad N(n) = \sum_{j=1}^{n-1} a_j \binom{n}{j} ,$$

where

$$(7) \qquad a_j = \sum_{\mathcal{O}_j} \binom{j}{\alpha_1}\binom{j-\alpha_1}{\alpha_2}\binom{j-(\alpha_1+\alpha_2)}{\alpha_3} \cdots \binom{\alpha_k}{\alpha_k} ,$$

$\mathcal{O}_j = \left\{A_{1k}\right\}$, the set of all <u>ordered</u> integral partitions of j,

$$A_{1k} = \left\{\alpha_{11}, \ldots, \alpha_{k1} \,\middle|\, \sum_{i=1}^{k} \alpha_{i1} = j\right\} \qquad k = 1, 2, \ldots, j.$$

To verify this formula it is only necesary to count, as we have already done for the terms $j=1$ and $j=2$, the identities for the j^{th} term. For each of these identities, the last operator to be applied is $D^{(j)}_{i_1 i_2 \ldots i_j}$, where the i_p form all the possible combinations of n integers taken j at a time. Hence there are $\binom{n}{j}$ choice for this operator. To complete an identity we must apply this operator to a combination of operators which contains each of the differentiations $\frac{\partial}{\partial x_{i_p}}$, $p = 1, 2, \ldots, j$, once and only once. Thus, for example, we may have a combination of j operators, each of which has one differentiation $\frac{\partial}{\partial x_{i_p}}$; or we may have a combination of $j-1$ operators, one of which contains the second order differentiation $\frac{\partial^2}{\partial x_{i_p} \partial x_{i_q}}$, while the others contain only first order differentiations $\frac{\partial}{\partial x_{i_p}}$; and so forth. The number of

[illegible]

choices for each operator can only be the number of combinations of the differentiations still to be included, taken as many at a time as the order of the operator being considered. Finally, since the order in which these operators are displayed is a factor in determining which identity we have, we must consider all ordered integral partitions of j. The summation in (6) is taken only up to $n-1$ since for $j=n$, we have the trivial identity $D(u)=D(u)$, and this is not to be counted in our determinations. If we tabulate the appropriate values for $n=2,3,4$, and 5, we have

$$\begin{array}{ll} a_1 = 1 & N(2) = 2 \\ a_2 = 3 & N(3) = 12 \\ a_3 = 13 & N(4) = 74 \\ a_4 = 75 & N(5) = 540 . \end{array}$$

D. We turn now to a consideration of the invariant character of the coefficients of an arbitrary R. Consider first the identities of the form

$$D^{(n-1)}_{1\,2\,\ldots\,\dot{i}-1\;\dot{i}+1\ldots\,n}\; D'_i\,(u) = D(u) + R.$$

The expression $D'_i\,(u)$ is of order $n-1$ and, in fact,

[illegible]

[illegible]

[illegible]

[illegible]

$$D_i'(u) = \frac{\partial^{(n-1)} u}{\partial x_1 \cdots \partial x_{i-1} \partial x_{i+1} \cdots \partial x_n} + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{1 \cdots j-1\, j+1 \cdots n} \frac{\partial^{(n-2)} u}{\partial x_1 \cdots \partial x_{i-1} \partial x_{i+1} \cdots \partial x_{j-1} \partial x_{j+1} \cdots \partial x_n} +$$

$$+ \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{1 \cdots j-1\, j+1 \cdots k-1\, k+1 \cdots n} \frac{\partial^{(n-3)} u}{\partial x_1 \cdots \partial x_{i-1} \partial x_{i+1} \cdots \partial x_{j-1} \partial x_{j+1} \cdots \partial x_{k-1} \partial x_{k+1} \cdots \partial x_n} +$$

$$+ \cdots + a_i u .$$

We may write this more compactly as

$$(8) \quad D_i'(u) = u_{(i)}^{(n-1)} + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{(j)} u_{(ij)}^{(n-2)} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{(jk)} u_{(ijk)}^{(n-3)} + \cdots + a_i u.$$

To illuminate the notation, the superscript on u indicates the order of differentiation, the subscript on u, which is in parentheses, indicates the variables with respect to which u is <u>not</u> differentiated and the subscripts of the a coefficients, which are in parentheses, indicate the integer or integers omitted from the subscripts of the a coefficients in D. For example,

$$a_{(j)} = a_{12 \cdots j-1\, j+1 \cdots n}$$

$$a_{(jk)} = a_{12 \cdots j-1\, j+1 \cdots k-1\, k+1 \cdots n}.$$

Observe that $a_{(jk)} = a_{(kj)}$, hence the factor $\frac{1}{2}$; for similar reasons, the q^{th} term in this expression contains the factor $\frac{1}{(q-1)!}$.

If we scrutinize equation (8) we find that there is just one term of order $n-1$, and its coefficient is 1; there are $n-1$ terms of order

$n-2$, and their coefficients are the $a_{(j)}$; there are $\binom{n-1}{2}$ terms of order $(n-3)$, with coefficients $a_{(jk)}$; and in general there are $\binom{n-1}{n-q-1}$ terms of order $n-q$. Consider now the effect of the operator $D^{(n-1)}_{1\ldots i-1\ i+1\ \ldots\ n} = \frac{\partial}{\partial x_i} + a_{(i)}$, when it is applied to $D_i'(u)$.

$$D^{(n-1)}_{1\ldots i-1\ i+1\ldots n} D_i'(u) = u^{(n)} + \sum_{j=1}^{n} a_{(j)} u^{(n-1)}_{(j)} + \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\frac{\partial a_{(j)}}{\partial x_i} + a_{(j)} a_{(i)} \right) u^{(n-2)}_{(ij)} +$$

$$(9) \qquad + \frac{1}{2} \sum_{\substack{j=1\\ j\neq i}}^{n} \sum_{\substack{k=1\\ k\neq i\\ j\neq k}}^{n} a_{(jk)} u^{(n-2)}_{(jk)} + \cdots + \frac{\partial a_i}{\partial x_i} u.$$

Equation (9) tells us immediately that there are no invariant coefficients for the terms of order $n-1$. This statement is readily seen to be true no matter which identity we consider. Thus a discussion of the invariants must begin with the terms of order $n-2$.

From the fact that the indices i, j, and k are distinct we conclude that $u_{(ij)} \neq u_{(jk)}$ for any j. We further note that (9) contains $(n-1) + \frac{(n-1)(n-2)}{2} = \binom{n}{2}$ different mixed derivatives of order $n-2$, and these are all the possible mixed derivatives of this order. Thus in order to obtain the identity

$$D^{(n-1)}_{1\ \ldots i-1\ i+1\ \ldots\ n} D_i'(u) = D(u) + R,$$

i.e. to make the expression $D(u)$ appear on the right hand side of (9), we must (at the very least) add and substract to the right hand side of (9) the terms $a_{(ij)} u^{(n-2)}_{(ij)}$. This results in the invariant coefficient of order $n-2$:

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

$$\frac{\partial a_{(j)}}{\partial x_i} + a_{(j)} a_{(i)} - a_{(ji)}, \quad i,j = 1,2,\ldots n, \; i \neq j. \tag{10}$$

Since there are n choices for i, and $n-1$ choices for j, we have a total of $n(n-1)$ invariant coefficients for the terms of order $n-2$. We will now prove the following:

Theorem V: The invariant coefficients for the terms $u_{(ij)}$ of order $n-2$ which appear in the remainder R of any identity are of the form

$$\frac{\partial a_{(j)}}{\partial x_i} + a_{(j)} a_{(i)} - a_{(ji)} .$$

These "invariants" are all true invariants, i.e. invariant under the change of variables $u = \lambda \bar{u}$, and no invariant coefficient for terms of order less than $n-2$ can, in general, be a "true invariant".

Consider any identity given by a sequence of operators,

$$D^{(a)}_{i_1 \ldots i_a} D^{(b)}_{j_1 \ldots j_b} \cdots D^{(q)}_{k_1 \ldots k_q} D^{(s)}_{l_1 \ldots l_s}(u) = D(u) + R.$$

where $(n-a) + (n-b) + \ldots + (n-s) = n.$

If we write this out in more detail, we find that such an identity is actually

$$\left[\frac{\partial^{(n-a)}}{\underset{(i \neq i_1, i_2, \ldots, i_a)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \sum_{j=1}^{n} a_{(j)} \frac{\partial^{(n-a-1)}}{\underset{(i,j \neq i_1, i_2, \ldots, i_a;\, i \neq j)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{n} a_{(jk)} \frac{\partial^{(n-a-2)}}{\underset{(i,j,k \neq i_1, \ldots, i_a;\, i \neq j,k)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \cdots \right] \circ$$

$$\circ \left[\frac{\partial^{(n-b)}}{\underset{(i \neq j_1, \ldots, j_b)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \sum_{j=1}^{n} a_{(j)} \frac{\partial^{(n-b-1)}}{\underset{(i,j \neq j_1 \cdots j_b;\, i \neq j)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \cdots \right] \circ \; . \; . \; . \circ$$

$$\circ \left[\frac{\partial^{(n-s)} u}{\underset{(i \neq l_1, \ldots, l_s)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \sum_{j=1}^{n} a_{(j)} \frac{\partial^{(n-s-1)} u}{\underset{(i,j \neq l_1, \cdots l_s;\, i \neq j)}{\partial x_1 \cdots \partial x_i \cdots \partial x_n}} + \cdots \right] = D(u) + R. \tag{11}$$

By an inspection of the left hand side of (11) we can determine how each term in $D(u)$ is made to appear on the right hand side of (11). To obtain the n^{th} order term $\frac{\partial^n u}{\partial x_1 \dots \partial x_n}$, we must combine the highest order differentiation in each operator $D^{(a)}_{i_1 \dots i_a}$, $D^{(b)}_{j_1 \dots j_b}, \dots, D^{(q)}_{k_1 \dots k_q}$, and apply the result to $\frac{\partial^{(n-S)} u}{\partial x_1 \dots \partial x_1 \dots \partial x_n}$ $(i \neq l_1 \dots, l_s)$. (The determination of this n^{th} order term is, in fact, the criteria used to determine which combination of operators yields an identity.) The terms of order $n-1$ which arise are obtained in two ways. They are obtained by combining the highest order differentiation in each operator $D^{(a)}_{i_1 \dots i_a}$, $D^{(b)}_{j_1 \dots j_b}, \dots, D^{(q)}_{k_1 \dots k_q}$, and applying the result to each term of the sum

$$\sum_{j\ 1}^{n} a_{(j)} \frac{\partial^{(n-S-1)} u}{\partial x_1 \dots \partial x_i \dots \partial x_n} \quad (i, j = l_1 \dots l_s;\ i \neq j) \tag{12}$$

Using Leibnitz's rule we see that, among others, the terms

$$a_{(j)}\ u^{(n-1)}_{(j)} \qquad \begin{array}{l} j = 1, 2, \dots, n \\ j \neq l_1 \dots l_s \end{array}$$

must appear. Such terms of order $(n-1)$ also arise when we combine, one by one, a differentiation of second highest order in any one operator $D^{(a)}_{i_1 \dots i_a}, \dots, D^{(q)}_{k_1 \dots k_q}$ with the highest order differentiation in every other operator $D^{(a)}_{i_1 \dots i_a}, \dots, D^{(q)}_{k_1 \dots k_q}$, and apply this result to

$$(13) \qquad \frac{\partial^{(n-s)} u}{\partial x_1 \cdots \partial x_i \cdots \partial x_n} \,. \qquad i \neq l_1 \ldots l_s$$

The coefficients of the terms of order $n-2$, which are the terms we are concerned with in this theorem, arise in four ways. We may obtain such terms by combining the highest order differentiation in each operator $D^{(a)}_{i_1 \ldots i_a}, \ldots, D^{(q)}_{k_1 \ldots k_q}$ and applying this result to each term of the sum (12). Again using Leibnitz's rule, we see that, among others, the terms

$$\frac{\partial a_{(j)}}{\partial x_i} \cdot u^{(n-2)}_{(ij)} \qquad i,j = 1,2,\ldots n;\ i \neq j; \quad i,j \neq l_1 \ldots l_s,$$

must appear. If we combine, one by one a differentiation of second highest order in any one operator $D^{(a)}_{i_1 \ldots i_a}, \ldots, D^{(q)}_{k_1 \ldots k_q}$, with the highest order differentiation in every other operator $D^{(a)}_{i_1 \ldots i_a}, \ldots, D^{(q)}_{k_1 \ldots k_q}$ and apply this result to each term of the sum (12), we see that, among others, the terms

$$a_{(j)}\, a_{(i)}\, u^{(n-2)}_{(ij)} \qquad i,j = 1,2,\ldots,n;\ i \neq j \quad i,j \neq l_1 \ldots l_s\,,$$

will arise. In a similar manner, if we combine a differentiation of third highest order in any one operator $D^{(a)}_{i_1 \ldots i_a}, \ldots D^{(q)}_{k_1 \ldots k_q}$ with the highest order differentiation in every other operator $D^{(a)}_{i_1 \ldots i_a}, \ldots D^{(q)}_{k_1 \ldots k_q}$ and apply this result to (13), we obtain the terms of the form

$$a_{(jk)}\, u^{(n-2)}_{(jk)}\,,$$

while if we combine the highest order differentiation of each operator

$D^{(a)}_{i_1\ldots i_a},\ldots D^{(q)}_{k_1\ldots k_q}$, and apply this result to each term of the sum

$$\sum_{\substack{j,k=1\\ j\neq k}}^{n} a_{(jk)} \frac{\partial^{(n-s-2)} u}{\partial x_1 \ldots \partial x_i \ldots \partial x_n} \qquad (i,j,k = 1_1 \ldots 1_s;\ i \neq j,k) ,$$

we must again obtain terms of the form

$$a_{(jk)}\, u^{(n-2)}_{(jk)} .$$

Comparing the coefficients of these terms of order $(n-2)$ and recalling the procedure necessary to make $D(u)$ appear on the right hand side of (9), we see immediately that the invariant coefficients for the terms $u^{(n-1)}_{(ij)}$, which appear in the remainder R must be of the prescribed form (10).

Knowing that these invariants must be of this form, we proceed to show that they are true invariants. We consider the equation $D(u) = 0$ and the change of variables $u = \lambda \bar{u}$, where $\lambda = \lambda(\bar{x})$ is not zero. Using the chain rule for differentiation of a product, and employing the notation of (8) we have

$$
(14)\quad
\begin{aligned}
u^{(n)} &= \lambda \bar{u}^{(n)} + \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} \bar{u}^{(n-1)}_{(i)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \lambda}{\partial x_i \partial x_j} \bar{u}^{(n-2)}_{(ij)} + \ldots \\
u^{(n-1}_{(i)} &= \lambda \bar{u}^{(n-1)}_{(i)} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\partial \lambda}{\partial x_j} \bar{u}^{(n-2)}_{(ij)} + \frac{1}{2} \sum_{\substack{j=1\\ j\neq i\\ j\neq k}}^{n} \sum_{\substack{k=1\\ k\neq i}}^{n} \frac{\partial^2 \lambda}{\partial x_j \partial x_k} \bar{u}^{(n-3)}_{(ijk)} + \ldots, \\
&\qquad i = 1,2,\ldots,n, \\
u^{(n-2)}_{(ij)} &= \lambda \bar{u}^{(n-2}_{(ij)} + \sum_{\substack{k=1\\ k\neq j\\ k\neq i}}^{n} \frac{\partial \lambda}{\partial x_k} u^{(n-3)}_{(ijk)} + \ldots, i,j = 1,2,\ldots,n,\ i \neq j ,
\end{aligned}
$$

and similar other expressions involving derivatives of order less than (n-2). Substituting the relations (14) into the equation $D(u) = 0$, and dividing through by λ, we obtain

$$\bar{D}(\bar{u}) = \bar{u}^{(n)} + \sum_{i=1}^{n} \left(a_{(i)} + \frac{\partial \log\lambda}{\partial x_i}\right) \bar{u}^{(n-1)}_{(i)} +$$

$$+ \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} \left(\frac{a_{(ij)}}{2} + a_{(i)} \frac{\partial \log\lambda}{\partial x_j} + \frac{1}{2\lambda} \frac{\partial^2 \lambda}{\partial x_i \partial x_j}\right) \bar{u}^{(n-2)}_{(ij)} + \dots = 0.$$

Let us indicate these new coefficients with the obvious notation

$$\bar{a}_{(i)} = a_{(i)} + \frac{\partial \log\lambda}{\partial x_i}$$

$$\bar{a}_{(ij)} = a_{(ij)} + a_{(i)} \frac{\partial \log\lambda}{\partial x_j} + a_{(j)} \frac{\partial \log\lambda}{\partial x_i} + \frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial x_i \partial x_j}.$$

Then the new invariant coefficients (associated with $\bar{D}$), for the terms $\bar{u}_{(ij)}$, are

$$\frac{\partial \bar{a}_{(i)}}{\partial x_j} + \bar{a}_{(j)}\, \bar{a}_{(i)} - \bar{a}_{(ij)},$$

and in terms of the original coefficients, these become

$$\frac{\partial a_{(i)}}{\partial x_j} + \frac{\partial^2 \log\lambda}{\partial x_i \partial x_j} + \left(a_{(i)} + \frac{\partial \log\lambda}{\partial x_i}\right)\left(a_{(j)} + \frac{\partial \log\lambda}{\partial x_j}\right) -$$

$$-\left(a_{(ij)} + a_{(i)} \frac{\partial \log\lambda}{\partial x_j} + a_{(j)} \frac{\partial \log\lambda}{\partial x_i} + \frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial x_i \partial x_j}\right)$$

$$= \left(\frac{\partial a_{(i)}}{\partial x_j} + a_{(i)}\, a_{(j)} - a_{(ij)}\right),$$ as was asserted.

To show that no invariant coefficient for terms of lesser order is a true invariant, we must first investigate further the character of these other invariants.

E. We will now give a general method for the determination of the invariant coefficient of any order term in an arbitrary remainder R. Consider any identity given by a sequence of operators.

$$D^{(a)}_{i_1\cdots i_a} D^{(b)}_{j_1\cdots j_b} \cdots D^{(q)}_{k_1\cdots k_q} D^{(s)}_{l_1\cdots l_s}(u) = D(u) + R,$$

where $(n-a) + (n-b) + \cdots + (n-s) = n.$

We will denote for brevity

$$D^{(a)}_{i_1\cdots i_a} \cdots D^{(q)}_{k_1\cdots k_q} = \mathcal{D}.$$

Suppose we wish to find the invariant coefficient for the derivative $u^{(p)}_{m_1\cdots m_p}$, when $c_o u^{(p)}_{m_1\cdots m_p}$ is a term in the expression $D^{(s)}_{l_1\cdots l_s}(u)$. Since each differentiation $\frac{\partial}{\partial x_i}$, $i = 1,\ldots n$, never appears in more than one operator on the left hand side of (15), we see that the left hand side of (15) contains the term

$$\mathcal{D}(c_o)\, u^{(p)}_{m_1\cdots m_p},$$

and no other term on the left hand side of (15) contains $u^{(p)}_{m_1\cdots m_p}$ as a factor. Thus, recalling the procedure necessary to make $D(u)$ appear on the right hand side of (15), we see that the invariant coefficient for $u^{(p)}_{m_1\cdots m_p}$ must be

(16) $$\mathcal{D}(c_o) - a_{m_1\cdots m_p}.$$

Now suppose we wish to find the invariant coefficient for the derivative $u^{(p)}_{m_1\ldots m_p}$, when there is no term in the expression $D^{(s)}_{l_1\ldots l_s}(u)$ which contains $u^{(p)}_{m_1\ldots m_p}$ as a factor. We may assume that $D^{(s)}_{l_1\ldots l_s}(u)$ contains a term $c_o\, u^{(p-j)}_{m_1\ldots m_{p-j}}$ for some $j = 1,2,\ldots p$, where we set $u^{(o)}_{m_o} \equiv u$. This statement is obviously true for, at worst, the expression $D^{(s)}_{l_1\ldots l_s}(u)$ must contain a term $c_o u$, (although c_o may vanish identically). We may then apply LeRoux's rule, extended to an operator with n independent variables, to the product $c_o u^{(p-j)}_{m_1\ldots m_{p-j}}$. Since the original operator D never contains any differentiation $\frac{\partial}{\partial x_i}$ multiply, for any $i = 1,2,\ldots, n$, the same is true of the operator $\mathcal{D}$. Thus our extension of LeRoux's rule for this type of operator is

$$(17)\quad D(uv) = u\,D(v) + \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} D'_j(v) + \frac{1}{2!} \sum_{\substack{j,k=1\\ j\neq k}}^{n} \frac{\partial^2 u}{\partial x_j \partial x_k} D''_{jk}(v) \quad \ldots,$$

and hence

$$(18)\quad \mathcal{D}(c_o u^{(p-j)}_{m_1\ldots m_{p-j}}) = \mathcal{D}(c_o) u^{(p-j)}_{m_1\ldots m_{p-j}} + \sum_{k=1}^{m-(p-j)} \mathcal{D}'_{m_{p-j+k}}(c_o) u^{(p-j+1)}_{m_1\ldots m_{p-j}\, m_{p-j+k}} +$$

$$+ \ldots + \mathcal{D}^{(j)}_{m_{p-j+1}\ldots m_p}(c_o) u^{(p)}_{m_1\ldots m_p} + \ldots \;.$$

From this we easily conclude that the invariant coefficient for $u^{(p)}_{m_1\ldots m_p}$ must be

$$(19)\qquad \mathcal{D}^{(j)}_{m_{p-j+1}\ldots m_p}(c_o) - a_{m_1\ldots m_p} \;.$$

We must point out that formula (19) does not give the complete picture. For if $\mathcal{D}^{(j)}_{m_{p-j+1}\cdots m_p}$ is the identity operator, then it must be that $c_0 = a_{m_1\cdots m_p}$, and that $a_{m_1\cdots m_p} u^{(p)}_{m_1\cdots m_p}$ appears on the left hand side of (15). That is to say, there is no need to add and substract $a_{m_1\cdots m_p} u^{(p)}_{m_1\cdots m_p}$ to the right hand side of (15), hence there is no term containing $u^{(p)}_{m_1\cdots m_p}$ in R. In other words, there will be no invariant coefficient for $u^{(p)}_{m_1\cdots m_p}$. The same conclusion holds whenever $a_{m_1\cdots m_p}$ appears as a coefficient in any one of the operators $D^{(a)}_{i_1\cdots i_a}$, $D^{(b)}_{j\cdots j_b}$, ..., $D^{(q)}_{k_1\cdots k_q}$. This is apparent from the left hand side equation (11), and the discussion which follows it regarding the determination of each term in D(u) which appears on the right hand side of (11). To summarize, whenever $a_{m_1\cdots m_p}$ appears as a coefficient in any operator $D^{(a)}_{i_1\cdots i_a}$, $D^{(b)}_{j_1\cdots j_b}$, ... $D^{(s)}_{l_1\cdots l_s}$, which is on the left hand side of (15), there will be no invariant coefficient for $u^{(p)}_{m_1\cdots m_p}$ appearing in R on the right hand side of (15). Otherwise, there will be an invariant coefficient (which may be zero) appearing in R on the right hand side of (15), which will be given by formula (16) or formula (19) as appropriate.

F. As yet we have said nothing about the character or number of quasi-invariants, i.e. the invariant coefficients for derivatives of u of order less than n-2. As we have pointed out, however, there will be no invariant coefficient for a derivative $u^{(p)}_{m_1\cdots m_p}$ when the coefficient $a_{m_1\cdots m_p}$ appears on the left hand side of any such identity (15). From this fact we may easily compute the maximum number of quasi-invariants associated with any

identity (15).

The total number of terms in $D(u)$ which are of order less than $n-2$, provided no coefficient is zero, is simply $\sum_{i=3}^{n}\binom{n}{i}$. If we subtract from this the total number of coefficients of derivatives of order less than $n-2$ in $D(u)$, which appear on the left hand side of (15), assuming all are not zero, we will have the maximum number of quasi-invariants which can appear in R, on the right hand side of (15). For if such a coefficient would appear on the left hand side of (15), were it not zero, then, regardless of the fact that it is zero, there will be no quasi-invariant corresponding to this term. This number, then, is

$$(20)\qquad \sum_{i=3}^{n}\binom{n}{i} - \left[\sum_{i=3}^{n-a}\binom{n-a}{i}+\sum_{i=3}^{n-b}\binom{n-b}{i}+\dots+\sum_{i=3}^{n-s}\binom{n-s}{i}\right],$$

$(n-a)+(n-b)+\dots+(n-s)=n$, where if $c<3$, we set $\sum_{i=3}^{c}\binom{c}{i}=0$.

For n 2, the number of quasi-invariants is obviously equal to zero. For $n=3$, we observe that an unique quasi-invariant appears as the coefficient of u in each identity, hence there are a total of 12 quasi-invariants for the entire third order case.

For $n>3$, however, the picture becomes somewhat clouded. This occurs because while no quasi-invariant can appear twice in the same identity, the same quasi-invariant can and does appear in more than one identity. Thus the determination of the total number of quasi-invariants for a given order n seems to be a rather tedious, lengthy, and unrewarding problem. To obtain an indication of the enormity of the task involved, however, it is quite simple to obtain upper and lower bounds on this number, which we shall designate $M(n)$.

An immediate lower bound is a consequence of the fact that an unique quasi-invariant results in each identity as the coefficient of u. Hence $N(n)$ is a lower bound for $M(n)$. To obtain an upper bound, we merely observe that the total number of quasi-invariants associated with D must be less than the number which would exist if every quasi-invariant appeared in only one identity. Now with every ordered integral partition of j,

$$A_{1k} = \left\{\alpha_{1l}, \alpha_{2l}, \dots \alpha_{kl} \;\middle|\; \sum_{i=1}^{k} \alpha_{il}\right\} = j \;,$$

there are associated

$$\binom{n}{j}\binom{j}{\alpha_1}\binom{j-\alpha_1}{\alpha_2}\binom{j-(\alpha_1+\alpha_2)}{\alpha_3} \cdots \binom{\alpha_k}{\alpha_k}$$

identities of the form

$$D^{(j)}\, D^{(n-\alpha_1)} D^{(n-\alpha_2)} \dots D^{(n-\alpha_k)}(u) = D(u) + R, \tag{21}$$

(see equations (6) and (7)).

With each of these identities are associated a maximum number of quasi-invariants, given by (20), namely

$$\sum_{i=3}^{n}\binom{n}{i} - \left[\sum_{i=3}^{n-j}\binom{j}{i} + \sum_{l=1}^{k}\sum_{i\,3}^{\alpha_l}\binom{n-\alpha_l}{i}\right] \tag{22}$$

Thus if we sum over $\mathfrak{A}_j$, the set of all ordered integral partitions of j, and then sum over j, we will obtain the number of quasi-invariants which would exist if every quasi-invariant appeared in only one identity. This number is

$$M^*(n) = \sum_{j=1}^{n-1} b_j(n)\binom{n}{j} \tag{23}$$

where

$$(24)\quad b_j(n) = \sum_{\alpha_j} \binom{j}{\alpha_1}\binom{j-\alpha_1}{\alpha_2} \cdots \binom{\alpha_k}{\alpha_k} \left\{ \sum_{i=3}^{n} \binom{n}{i} - \left[\sum_{i=3}^{n-j} \binom{j}{i} + \sum_{l=1}^{k} \sum_{i=3}^{\alpha_k} \binom{n-\alpha_k}{i} \right] \right\}.$$

Then for $n \geqq 3$, we have

$$N(n) \leqq M(n) \leqq M^*(n).$$

Computing a few values for $M^*(n)$ we find the following:

n	$N(n)$	$M^*(n)$
3	12	12
4	74	362
5	540	8510 .

G. Now we are in a position to complete the proof of Theorem V. We must show that no invariant coefficient for terms of order less than $n-2$ can in general be a true invariant. We shall prove this rigorously only for identities in which $\mathcal{D} = \frac{\partial}{\partial x_r} + a_{(r)}$, and shall only indicate the proof for other identities.

Consider the invariant coefficient for $u_{m_1 \ldots m_p}$, where $p < n-2$, and $m_i \neq r$ for any $i = 1, 2, \ldots, p$. Then $\sum_{i=0}^{p} c_i u_{m_1 \ldots m_{p-i}}$ is contained in $D_r'(u)$ and in fact

$$(25)\quad \begin{aligned} c_0 &= a_{m_1 \ldots m_p r}, \\ c_1 &= a_{m_1 \ldots m_{p-1} r}, \\ c_p &= a_r, \end{aligned}$$

where of course r must be properly ordered with $m_1, \ldots, m_p$.

Now consider the change of variables $u = \lambda \bar{u}$, where $\lambda = \lambda(\bar{x}) \not\equiv 0$.
We extend the equations (14) to obtain

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} = \cdots + \frac{\partial^{n-p}\lambda}{\partial x_{m_{p+1}} \cdots \partial x_{m_n}} \bar{u}_{m_1 \cdots m_p} + \cdots$$

$$\frac{\partial^{n-1} u}{\partial x_1 \cdots \partial x_{i-1} \partial x_{i+1} \cdots \partial x_n} = \cdots \frac{\partial^{n-p-1}\lambda}{\partial x_{m_{p+1}} \cdots x_{m_{p+i-1}} \cdots x_{m_{p+i+1}} \cdots \partial x_{m_n}} \bar{u}_{m_1 \cdots m_p} \cdots \tag{26}$$

$$\cdots \qquad\qquad 0 \leqq i \leqq n-p$$

$$\frac{\partial^p u}{\partial x_{m_1} \cdots \partial x_{m_p}} = \lambda \bar{u}_{m_1 \cdots m_p} + \cdots \quad .$$

If we substitute (26) into $D(u) = 0$, and divide through by λ, we obtain $\bar{D}(\bar{u})$. The coefficients in which we are interested are then

$$\bar{a}_{m_1 \cdots m_p} = a_{m_1 \cdots m_p} + \sum_{j=p+1}^{n} a_{m_1 \cdots m_p m_j} \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_{m_j}} +$$

$$+ \sum_{\substack{j,k=p+1 \\ j \neq k}}^{n} a_{m_1 \cdots m_p m_j m_k} \frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial x_{m_j} \partial x_{m_k}} + \qquad \cdots \quad \cdots \quad \cdots \tag{27}$$

$$+ \sum_{\substack{j,k,l=p+1 \\ j \neq k \neq l \neq j}}^{n} a_{m_1 \cdots m_p m_j m_k m_l} \frac{1}{\lambda} \frac{\partial^3 \lambda}{\partial x_{m_j} \partial x_{m_l} \partial x_{m_k}} +$$

$$+ \cdots + \frac{1}{\lambda} \frac{\partial^{n-p}\lambda}{\partial x_{m_{p+1}} \cdots \partial x_{m_n}} ;$$

$$\bar{a}_{m_1\ldots m_p r} = a_{m_1\ldots m_p r} + \sum_{\substack{j=p+1 \\ j\neq\rho}}^{n} a_{m_1\ldots m_p m_j r}\ \frac{1}{\lambda}\frac{\partial\lambda}{\partial x_{m_j}} +$$

$$+\sum_{\substack{j,k=p+1 \\ j,k\neq\rho \\ j\neq k}}^{n} a_{m_1\ldots m_p m_j m_k r}\ \frac{1}{\lambda}\frac{\partial^2\lambda}{\partial x_j\,\partial x_k} + \cdots +$$

$$+\frac{1}{\lambda}\,\frac{\partial^{\,n-(p+1)}\lambda}{\partial x_{m_{p+1}}\cdots\partial x_{m_{\rho-1}}\,\partial x_{m_{\rho+1}}\cdots\partial x_{m_n}}\,, \quad \text{where } x_{m_\rho} = x_r\ ;$$

and $\bar{a}_{(r)} = a_{(r)} + \frac{1}{\lambda}\frac{\partial\lambda}{\partial x_r}$.

In all the above we must comment that the subscripts involved must all be properly ordered in accordance with the conditions of (5).

The quasi-invariant for $u_{m_1\ldots m_p}$ in this particular identity is simply $Q = \mathcal{O}(c_o) - a_{m_1\ldots m_p}$, or

$$Q = \frac{\partial}{\partial x_r}(a_{m_1\ldots m_p r}) + a_{(r)}(a_{m_1\ldots m_p r}) - a_{m_1\ldots m_p}\,.$$

Then the transformed quasi-invariant, for $\bar{u}_{m_1\ldots m_p}$, will be

$$(28)\qquad \bar{Q} = \frac{\partial}{\partial x_r}(\bar{a}_{m_1\ldots m_p r}) + \bar{a}_{(r)}(\bar{a}_{m_1\ldots m_p r}) - \bar{a}_{m_1\ldots m_p}\,.$$

Substituting relations (27), we find, after a somewhat tedious computation that

$$\overline{Q} = Q + \sum_{\substack{j=p+1 \\ j\neq p+\varrho}}^{n} \left(\frac{\partial}{\partial x_r} (a_{m_1\ldots m_p m_j r}) + a_{(r)} a_{m_1\ldots m_p m_j r} - a_{m_1\ldots m_p m_j} \right) \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_{m_j}} +$$

$$+ \sum_{\substack{j,k=p+1 \\ j\neq k \\ j,k\neq p+\varrho}}^{n} \left(\frac{\partial}{\partial x_r} (a_{m_1\ldots m_p m_j m_k r}) + a_{(r)} a_{m_1\ldots m_p m_j m_k r} - a_{m_1\ldots m_p m_j m_k} \right) \frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial x_{m_j} \partial x_{m_k}} +$$

(29)

$$+ \ldots +$$

$$+ \sum_{\substack{j=p+1 \\ j\neq p+\varrho}}^{n} \left(\frac{\partial}{\partial x_r} (a_{(j)}) + a_{(r)} a_{(j)} - a_{(jr)} \right) \frac{\partial^{n-(p+2)} \lambda}{\partial x_{m_{p+1}} \ldots \partial x_{m_{j-1}} \partial x_{m_{j+1}} \ldots \partial x_{m_{p+\varrho-1}} \partial x_{m_{p+\varrho+1}} \ldots \partial x_{m_n}} .$$

We observe that the coefficient of each derivative of λ in this expression is itself a quasi-invariant which arises from the particular identity we have chosen.

Then using an obvious notation we write (29) as

$$\overline{Q} = Q + \sum Q_j \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_{m_j}} + \sum Q_{jk} \frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial x_{m_j} \partial x_{m_k}} +$$

(30)

$$+ \sum Q_{jkl} \frac{1}{\lambda} \frac{\partial^3 \lambda}{\partial x_{m_j} \partial x_{m_k} \partial x_{m_l}} + \ldots + \sum Q_{(jr)} \frac{1}{\lambda} \lambda^{(n-(p+2))}_{(jr)} .$$

Now if $p = n-2$, then $\overline{Q} = Q + Q_n \frac{1}{\lambda} \frac{\partial \lambda}{\partial x_n}$.

But $Q_n = \frac{\partial}{\partial x_r}(a_{1\ldots n}) + a_{(r)}a_{1\ldots n} - a_{1\ldots(r-1)(r+1)\ldots n} =$

$$= \frac{\partial}{\partial x_r}(1) + a_{(r)}\cdot 1 - a_{(r)} = 0, \text{ hence}$$

$\overline{Q} = Q$ as was proved previously in the more general case.

If $p > n-2$, however, then $\overline{Q} = Q +$ a linear combination of other invariants arising from the same identity. Thus $\overline{Q} \neq Q$ in general, for the particular identities $D^{(n-1)}_{x_1\ldots x_{r-1}\, x_{r+1}\ldots x_n}\, D'_{x_r}(u) = D(u) + R.$

Although we shall not prove it rigorously here, it easy to see that for $p > n-2$, no matter which identity we choose, terms involving $\frac{\partial\lambda}{\partial x_{m_j}}$, $\frac{\partial^2\lambda}{\partial x_{m_j}\partial x_{m_k}}$, ..., will arise in the expression for any transformed quasi-invariant, since these factors appear in the transformed equation $\overline{D}(\overline{u}) = 0$. Thus, we assert that for any quasi-invariant Q, $\overline{Q} \neq Q$, and in fact we may expect that $\overline{Q} = Q +$ a linear combination of other invariants arising from the same identity. This is the justification for the name "quasi-invariant, and this completes the proof of Theorem V.

H. In order to see the problems involved in applying the cascade method for the n^{th} order operators we have been considering, let us consider a particular kind of identity once again. We will study identities of the form

(31) $\quad D'_r\, D^{(n-1)}_{1\ldots r-1\; r+1\ldots n}(u) = D'_r\left(\frac{\partial u}{\partial x_r} + a_{(r)}(u)\right) =$

$$= D(u) + \sum_{\substack{j=1 \\ j\neq r}}^{n} \left(\frac{\partial a_{(r)}}{\partial x_j} + a_{(r)} a_{(j)} - a_{(jr)} \right) u_{(jr)} + \sum Q_{i_1 \dots i_{n-3}} u_{i_1 \dots i_{n-3}} , \tag{31}$$

where $Q_{i_1 \dots i_{n-3}}$ is the quasi-invariant associated with $u_{i_1 \dots i_{n-3}}$, $i_p \neq r$, $p = 1, \dots n-3$.

Denote $D^{(n-1)}_{1 \dots r-1 r+1 \dots n}(u) = \frac{\partial u}{\partial x_r} + a_{(r)} u \equiv u'$, $T_j \equiv \equiv \frac{\partial a_{(r)}}{\partial x_j} + a_{(r)} a_{(j)} - a_{(jr)}$, and observing that $D(u) = 0$, we write

$$u' = \frac{\partial u}{\partial x_r} + a_{(r)} (u) , \tag{32}$$

$$D_r'(u') = \sum_{\substack{j=1 \\ j\neq r}}^{n} T_j u_{(jr)} + \sum Q_{i_1 \dots i_{n-3}} u_{i_1 \dots i_{n-3}} .$$

It would be most desirable if all the T's and all the Q's in (32) were identically zero, for if this were true we would have succeeded in reducing the order of the original equation by one, resulting in a single first order equation, coupled with an equation of order $(n-1)$. Ideally by this method we would like to reduce the n^{th} order equation to n first order equations, but this happy result will, unfortunately, occur very seldom. In the more probable event that not all the T's and Q's vanish identically, we would then hope at least to be able to cascade the equations in the manner of the preceding chapters, until such time as the T's and the Q's may all be identically zero. To this end it will obviously be necessary to impose certain rather rigid conditions upon the coefficients of $D(u)$. Let the hypotheses be

(a) $T_1 \equiv \ldots T_{r-1} \equiv T_{r+1} \equiv \ldots \equiv T_n \equiv Q_{i_1 \ldots i_{n-3}} \equiv T \not\equiv 0$,

(b) all coefficients are functions of x_r only,

(c) all coefficients with an "r" subscript are identical.

Then

$$D_r'(u') = T \sum u_{i_1 \ldots i_{n-2}} \quad , \quad i_p \neq r. \tag{33}$$

Solving $u' = \frac{\partial u}{\partial x_r} + a_{(r)} u$, for u, we find

$$u = e^{-\int a_{(r)} dx_r} \left\{ \int e^{\int a_{(r)} dx_r} u' \, dx_r + F(x_1 \ldots x_{r-1}, x_{r+1}, \ldots x_n) \right\},$$

hence

$$u_{i_1 \ldots i_{n-2}} = \frac{\partial^q}{\partial x_{i_1} \ldots \partial x_{i_{n-2}}} \left[e^{-\int a_{(r)} dx_r} \left\{ \int e^{\int a_{(r)} dx_r} u' dx_r + F \right\} \right] =$$

$$= e^{-\int a_{(r)} dx_r} \frac{\partial^q}{\partial x_{i_1} \ldots \partial x_{i_{n-2}}} \left\{ \int e^{\int a_{(r)} dx_r} u' dx_r + F \right\},$$

where q is the number of non-zero i_p.

Substituting these expressions into (33) we obtain

$$\frac{e^{\int a_{(r)} dx_r}}{T} D_r'(u') = \sum \frac{\partial^q}{\partial x_{i_1} \ldots \partial x_{i_{n-2}}} \left\{ \int e^{\int a_{(r)} dx_r} u' dx_r + F \right\}, \quad i_p \neq r.$$

We then take the partial derivative of both sides of this expression with respect to x_r, to obtain

$$\frac{e^{\int a_{(r)} dx_r}}{T} \left[a_{(r)} D_r'(u') + \frac{\partial}{\partial x_r} (D_r'(u')) - \frac{\partial \log T}{\partial x_r} D_r'(u') \right] =$$

$$= e^{\int a_{(r)} dx_r} \sum \frac{\partial^q u'}{\partial x_{i_1} \ldots \partial x_{i_{n-2}}} , \quad i_p \neq r .$$

Cancelling out the exponentials and collecting terms we find that we have an expression

$${}_1D\,(u') = 0\,,$$

where ${}_1D$ is an n^{th} order linear operator of the form (5), but with coefficients differing from D. These coefficients we may compute to be

$$a'_{(r)} = a_{(r)} - \frac{\partial \log T}{\partial x_r}\,,$$

$$a'_{i_1 \ldots i_n} = a_{i_1 \ldots i_n} \qquad \text{where } i_p = r \text{ for some } p = 1, \ldots n,$$

$$a'_{i_1 \ldots i_{n-2}} = \frac{\partial a_{i_1 \ldots i_{n-2}}{}^{r}}{\partial x_r} + a_{i_1 \ldots i_{n-2}}{}^{r} \cdot a'_{(r)} - T\,, \text{ where}$$

$i_p \neq r$ for any $p = 1, \ldots, n-2$.

If now we compute the new invariants we find

$$T'_j = T - \frac{\partial a_{(j)}}{\partial x_r} \qquad j = 1, 2, \ldots r-1,\ r+1, \ldots n$$

and $$Q'_{i_1 \ldots i_{n-3}} = T - \frac{\partial a_{i_1 \ldots i_{n-3}}{}^{r}}{\partial x_r}\,.$$

But by hypothesis (c), $a_{(j)} = a_{i_1 \ldots i_{n-3}}{}^{r}$ for all $j \neq r$ and all permutations $i_1 \ldots i_{n-3}$, with $i_p \neq r$ for any $p = 1, \ldots, n-3$.

Thus if the new invariants are all zero, we may reduce the order of the equation ${}_1D(u') = 0$ by one. If the new invariants are not all zero, the hypotheses of this method are again satisfied, and we may cascade the equations in the hope that the invariants will eventually become all zero. This can only happen if $T = n \dfrac{\partial a_{(j)}}{\partial x_r}$, where n is some positive integer.

Other methods can be developed for identities of this form and for identities of other forms, but in every case, severe restrictions must be placed on the coefficients before cascading may begin. We see then that the problem of attacking these equations in this manner becomes extremely difficult and restrictive as n increases. For $n > 4$ the problem becomes enormous, with staggering numbers of invariants and identities to be manipulated, and a large number of restrictions on the coefficients necessary to cascade the equations. If it is desired to cascade the equations it is recommended that only first order substitutions be employed.

SECTION VI

A BRIEF SUMMARY OF EARLIER EXTENSIONS

A. Extension of the Laplace cascade method have been made by many notable mathematicians. One of the earliest of such extensions was made by Darboux himself.[21] Darboux considers the following system of equations of second order, with n independent variables.

Let $\rho_0, \rho_1, \rho_2, \ldots, \rho_{n-1}$ be a system of n independent variables. and consider the system of equations

$$(1) \qquad \frac{\partial^2 u}{\partial\rho_i \partial\rho_k} = a_{ik}\frac{\partial u}{\partial\rho_k} + a_{ki}\frac{\partial u}{\partial\rho_i} \quad ; \; i,k=0,1,2,\ldots n-1; \; i \neq k .$$

This system contains $\frac{n(n-1)}{2}$ linear hyperbolic equations of second order, and we observe that this is an overdetermined system, whereas the system considered by us in Section III is exactly determined. Darboux seeks first a necessary condition to give n linearly independent solutions in addition to the trivial solution $u = \text{constant}$. To accomplish this, Darboux forms the third derivative by taking $\frac{\partial}{\partial\rho_l}$ of both sides of (1). Interchanging the indices l and k, and equating coefficients of like derivatives of u, we obtain the relations

$$(2) \qquad \frac{\partial a_{ik}}{\partial\rho_l} = a_{i}\, a_{k} + a_{li}a_{ik} - a_{ik}a_{lk} \qquad (l = k \neq i).$$

Interchanging the indices i and l does not change the right hand side of (2) and hence

$$\frac{\partial a_{ik}}{\partial\rho_l} = \frac{\partial a_{lk}}{\partial\rho_i} .$$

Holding k constant, we see from integrability conditions that there must exist a function, call it $\log H_k$, such that

$$(3) \qquad a_{ik} = \frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}, \qquad (i \neq k).$$

Then condition (2) becomes

$$(4) \qquad \frac{\partial^2 H_k}{\partial \rho_i \partial \rho_l} = \frac{1}{H_i}\frac{\partial H_i}{\partial \rho_l}\frac{\partial H_k}{\partial \rho_i} + \frac{1}{H_l}\frac{\partial H_l}{\partial \rho_i}\frac{\partial H_k}{\partial \rho_l},$$

and the system (1) has the form

$$(5) \qquad \frac{\partial^2 u}{\partial \rho_i \partial \rho_k} - \frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}\frac{\partial u}{\partial \rho_k} - \frac{1}{H_i}\frac{\partial H_i}{\partial \rho_k}\frac{\partial u}{\partial \rho_i} = 0,$$

$$i,k = 0,1,2,\ldots,n-1;\ i \neq k.$$

Let $$f_{ik}(u) = \frac{\partial^2 u}{\partial \rho_i \partial \rho_k} - \frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}\frac{\partial u}{\partial \rho_k} - \frac{1}{H_i}\frac{\partial H_i}{\partial \rho_k}\frac{\partial u}{\partial \rho_i}.$$

Using condition (4) it is easy to verify that

$$(6) \qquad f_{ik}(H_l) = 0, \quad i \neq k \neq l.$$

To start the Laplace method we assume that we have an equation of the form (5), satisfying the conditions of integrability (4). Thus we consider the particular equation (for fixed i, k),

$$(7) \qquad f_{ik}(u) = 0.$$

Darboux then considers the substitution function ν defined by the relation

$$(8) \qquad \frac{\partial u}{\partial \rho_i} = \frac{1}{H_k}\frac{\partial H_k(u+\nu)}{\partial \rho_i}.$$

If we substitute (8) into (7), the equation then becomes

$$(9) \quad \frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}\frac{\partial \nu}{\partial \rho_k} + \left(\frac{\partial^2 \log H_k}{\partial \rho_i \partial \rho_k} - \frac{1}{H_i}\frac{\partial H_i}{\partial \rho_k}\,\frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}\right)(u+\nu) = 0 .$$

We see that our invariant h_{ik} is now

$$h_{ik} = \left(\frac{\partial^2 \log H_k}{\partial \rho_i \partial \rho_k} - \frac{1}{H_i}\frac{\partial H_i}{\partial \rho_k}\,\frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}\right),$$

and if $h_{ik} = 0$, the equation (9) has the simple solution

$$\nu = \text{constant} .$$

We may gain a little insight into these proceedings if we adopt the notation of Section V. Thus, for equation (7),

$$D_{ik} = \frac{\partial^2}{\partial \rho_i \partial \rho_k} - \frac{1}{H_k}\frac{\partial H_k}{\partial \rho_i}\frac{\partial}{\partial \rho_k} - \frac{1}{H_i}\frac{\partial H_i}{\partial \rho_k}\frac{\partial}{\partial \rho_i} =$$

$$= \frac{\partial^2}{\partial \rho_i \partial \rho_k} - a_{ik}\frac{\partial}{\partial \rho_k} - a_{ki}\frac{\partial}{\partial \rho_i},$$

and

$$D'_{ik\,\rho_i} = \frac{\partial}{\partial \rho_k} - a_{ki} , \qquad D'_{ik\,\rho_k} = \frac{\partial}{\partial \rho_i} - a_{ik} .$$

Then the substitution function ν is defined by

$$(10) \qquad D'_{ik\,\rho_k}(u) - a_{ik}\nu = 0 ,$$

while the invariant h_{ik} may be written

(11) $$h_{ik} = D'_{ik\rho_i}(a_{ik}),$$

and hence (9) becomes

(12) $$D'_{ik\rho_i}\left(D'_{ik\rho_k}(u)\right) = D_{ik}(u) + \left(D_{ik\rho_i}(-a_{ik})\right)u \; .$$

If $D'_{ik\rho_i}(-a_{ik}) = 0$, we see that (10) reduces the order of (12) by one, and the equation can be solved by quadratures.

In the more likely event that $D'_{ik\rho_i}(-a_{ik}) \not\equiv 0$, we may cascade the equations in the following manner. Let

$$L_k = -\frac{H_k}{\frac{\partial H_k}{\partial \rho_i}} D_{ik}(H_k)$$

and

$$L_i = \frac{H_i H_k}{\frac{\partial H_k}{\partial \rho_i}} \; .$$

Then

$$\frac{\partial \nu}{\partial \rho_k} = \frac{L_k}{H_k}(u+\nu),$$

and ν satisfies the equation

(13) $$\frac{\partial^2 \nu}{\partial \rho_i \partial \rho_k} - \frac{1}{L_k}\frac{\partial L_k}{\partial \rho_i}\frac{\partial \nu}{\partial \rho_k} - \frac{1}{L_i}\frac{\partial L_i}{\partial \rho_k}\frac{\partial \nu}{\partial \rho_i} = 0 .$$

In like manner, if we introduce the quantities L_k' defined by the relations

$$L_{k}' = \frac{H_k \dfrac{\partial H_{k}'}{\partial \rho_i} - H_k' \dfrac{\partial H_k}{\partial \rho_i}}{\dfrac{\partial H_k}{\partial \rho_i}}, \qquad (k' \neq i \neq k),$$

then the function ν, for distinct values of i' and k', satisfies the system of equations

$$(14) \qquad \frac{\partial^2 \nu}{\partial \rho_{i'} \partial \rho_{k'}} = \frac{1}{L_{i'}} \frac{\partial L_{i}}{\partial \rho_{k'}} \frac{\partial \nu}{\partial \rho_{i'}} + \frac{1}{L_{k'}} \frac{\partial L_{k'}}{\partial \rho_{i'}} \frac{\partial \nu}{\partial \rho_{k'}},$$

which is of the same form as (5). Hence we may iterate the substituion, and cascade the equations until such time as the corresponding invariants may vanish.

B. In 1899, J. Le Roux[22] extended the method of Laplace to linear partial differential equations of greater order than second. Le Roux, however considered the equation with only two independent variables, while we, in Section V, have considered the equation in n independent variables. In the notation of Le Roux, the equation to be considered is the n^{th} order equation

$$(15) \qquad D(z) = \sum \frac{n!}{\alpha!\,\beta!\,(n-\alpha-\beta)!} A_{\alpha,\beta} \frac{\partial^{\alpha+\beta} z}{\partial x^{\alpha} \partial y^{\beta}} = 0,$$

where $\alpha+\beta \leqq n$, $\alpha \neq n$, $\beta \neq n$.

If we suppose that the highest order of differentiation with respect to x in equation (15) is equal to $n-p$, then the collection of terms which contain such a differentiation are

$$a \frac{\partial^{n-p+i} z}{\partial x^{n-p} \partial y^{i}} + b \frac{\partial^{n-p+i-1} z}{\partial x^{n-p} \partial y^{i-1}} + \cdots + g \frac{\partial^{n-p} z}{\partial x^{n-p}} .$$

We denote the expression

$$a\frac{\partial^{i}}{\partial y^{i}}+b\frac{\partial^{i-1}}{\partial y^{i-1}}+\quad \dots +g$$

as the differential multiplier of the term $\frac{\partial^{n-p}z}{\partial x^{n-p}}$, and we further assume that the coefficient a is equal to one. Le Roux considers the equation (15) from the point of view that there exists a particular integral of the form of Euler

$$(16)\qquad z = u_o X^{(m)} + u_1 X^{(m-1)} + \dots + u_m X.$$

where X is an "arbitrary" function of x, and the coefficients u are functions of x and y.

If we regard $D(z)$ as a polynomial in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, we may introduce the notation of Section V. We designate $D^{(p\ q)}_{x^p y^q}(z)$ as the expression obtained when we differentiate D with respect to $\frac{\partial}{\partial x}$, p times, with respect to $\frac{\partial}{\partial y}$, q times, and apply this operator to the variable Z. For instance, the $(p+q)^{th}$ derivative of the term

$$A_{\alpha\beta}\frac{\partial^{\alpha+\beta}z}{\partial x^{\alpha}\partial y^{\beta}}$$

is

$$\alpha(\alpha-1)\cdots(\alpha-p+1)\,\beta(\beta-1)\cdots(\beta-q+1)\,A_{\alpha,\beta}\frac{\partial^{\alpha-p+\beta-q}z}{\partial x^{\alpha-p}\partial y^{\beta-q}},$$

and hence

$$(17)\qquad D^{(p+q)}_{x^p y^q}(z) = n(n-1)\dots(n-p-q+1)\sum\frac{(n-p-q)!}{(\alpha-p)!\,(\beta-q)!\,(n-\alpha-\beta)!}\,A_{\alpha,\beta}\frac{\partial^{\alpha-p+\beta-q}z}{\partial x^{\alpha-p}\partial y^{\beta-q}}.$$

Since (15) is assumed to have order $(n-p)$ with respect to x, we see that the coefficients of (16) must satisfy the equations

$$\frac{1}{(n-p)!} D_x^{(n-p)}{}_{n-p} (u_0) = 0$$

(18)
$$\frac{1}{(n-p)!} D_x^{(n-p)}{}_{n-p} (v_1) + \frac{1}{(n-p-1)!} D_x^{(n-p-1)}{}_{n-p-1} (u_0) = 0 \ldots$$

$$\cdots$$

$$\frac{1}{(n-p)!} D_x^{(n-p)}{}_{n-p} (u_i) + \frac{1}{(n-p-1)!} D_x^{(n-p-1)}{}_{n-p-1} (u_{i-1}) + \ldots = 0$$

$$\cdots .$$

In the case when the variable x is a simple characteristic variable, Le Roux considers the transformation

$$z_1 = \frac{1}{(n-1)!} D_x^{(n-1)}{}_{n-1} (z) = \frac{\partial z}{\partial y} + nA_{n-1,0}\, z,$$

and defines the functions ν and ν_1,

$$\nu = \frac{z}{u_0} = ze^{\int nA_{n-1} dy}$$

(19)
$$\nu_1 = \frac{z_1}{u_0} = z_1 e^{\int nA_{n-1} dy} \;;\; \nu_1 = \frac{\partial \nu}{\partial y} .$$

We then consider the equation resulting from (15) by means of the transformation (19), and collecting on the right all terms which do not contain any differentiation with respect to y, we obtain

(20)
$$\Delta(\nu_1) = \lambda_0 \frac{\partial^p \nu}{\partial x^p} + \lambda_1 \frac{\partial^{p-1} \nu}{\partial x^{p-1}} + \lambda_2 \frac{\partial^{p-2} \nu}{\partial x^{p-2}} + \ldots + \lambda_p \nu,$$

where $\Delta(\nu_1)$ designates a differential expression of order $(n-1)$, in which the coefficient of $\partial^{n-1} \nu_1 / \partial x^{n-1}$ is equal to unity. The coefficients are given by the formula.

(21)
$$\lambda_i = e^{\int nA_{n-1} dy} \frac{1}{(p-i)!} D_x^{(p-1)}{}_{p-1} (u_0) = \frac{1}{u_0} \frac{1}{(p-i)!} D_x^{(p-i)}{}_{p-i} (u_0) .$$

The order p of the right hand side of (20) is, in general, equal to $(n-2)$. It will be less only if

$$D^{(n-2)}_{x^{n-2}}(u_0) = 0 .$$

Le Roux then proceeds to define the invariants h, by the following rule. If the λ's regarded as functions of y are all linearly independent, then

$$(22) \qquad h_i = \begin{vmatrix} \lambda_0 & \lambda_1 & \dots \lambda_i \\ \frac{\partial \lambda_0}{\partial y} & \frac{\partial \lambda_1}{\partial y} & \dots \frac{\partial \lambda_i}{\partial y} \\ \dots & \dots & \dots \\ \frac{\partial^i \lambda_0}{\partial y^i} & \frac{\partial^i \lambda_1}{\partial y^i} & \dots \frac{\partial^i \lambda_i}{\partial y^i} \end{vmatrix}$$

which are $(p+1)$ in number, setting $h_0 = \lambda_0$. If the λ's, regarded as functions of y, are not linearly independent, we define λ_{α_1} as the first λ which is linearly independent of λ_0, λ_2 as the first λ which is linearly independent of λ_0 and λ_1, etc. Then the determinant (22) is replaced by

$$(22') \quad h_i = \begin{vmatrix} \lambda_0 & \lambda_{\alpha_1} \cdots & \lambda_{\alpha_i} \\ \frac{\partial \lambda_0}{\partial y} & \frac{\partial \lambda_{\alpha_1}}{\partial y} & \frac{\partial \lambda_{\alpha_i}}{\partial y} \\ \dots & \dots & \dots \\ \frac{\partial^i \lambda_0}{\partial y^i} & \frac{\partial^i \lambda_{\alpha_1}}{\partial y^i} & \frac{\partial^i \lambda_{\alpha_i}}{\partial y^i} \end{vmatrix} .$$

To show that these are generalizations of the Darboux invariants, we must consider the transformation

$$(23) \qquad \begin{aligned} z &= z' \; f(x,y), \\ &\begin{cases} x = \varphi(x') \\ y = \psi(x',y') . \end{cases} \end{aligned}$$

The first of these transformations is effected without changing the value of ν, since $u_0' = \frac{u_0}{f(x,y)}$, and hence

$$\frac{z'}{u_0'} = \frac{z}{u_0}.$$

The transformation of the independent variables, however, causes equation (20) to become

$$(24)\quad \Delta_1(\nu_1') = \lambda_0' \frac{\partial^p \nu}{\partial x'^p} + \lambda_1' \frac{\partial^{p-1}\nu}{\partial x'^{p-1}} + \dots + \lambda_p' \nu$$

where

$$\nu_1' = \nu_1 \frac{\partial y}{\partial y'}$$

( )

and

$$\lambda_0' = \lambda_0 \left(\frac{\partial x}{\partial x'}\right)^{n-1-p} \frac{\partial y}{\partial y'}$$

$$(25)\quad \lambda_1' = \frac{\partial y}{\partial y'} \left(\frac{\partial x}{\partial x'}\right)^{n-1-p+1} (\lambda_1 + \Theta_{10}\lambda_0)$$

$$\lambda_2' = \frac{\partial y}{\partial y'} \left(\frac{\partial x}{\partial x'}\right)^{(n-1-p+2)} (\lambda_2 + \Theta_{21}\lambda_1 + \Theta_{20}\lambda_0)$$

$$\dots$$

the coefficients Θ being dependent only on x. If we use the relations (25) to compute the new invariants h_i', we will find that

$$h_i' = \left(\frac{\partial y}{\partial y'} \frac{\partial x}{\partial x'}\right)^{\frac{(i+1)(i+2)}{2}} \frac{\partial x}{\partial x'}^{(n-2-p)(i+1)} h_i$$

Since in general $p = n-2$, the above relation reduces to

$$h_i' = \left(\frac{\partial y}{\partial y'} \frac{\partial x}{\partial x'}\right)^{\frac{(i+1)(i+2)}{2}} h_i .$$

Thus the determinant h_i is reproduced mutliplied by the $\frac{(i+1)\ (i+2)}{2}$

power of the Jacobian of the transformation. This justifies the name "invariant" for the determinants h_i, and LeRoux calls these the first generalization of the Darboux invariants.

Now for the equation (15) to admit an integral of the form $z = u_o X$, X being an arbitrary function of x only, it is necessary and sufficient that the coefficients λ of (20) are all zero. This is evident if we but consider

$$D(u_o X) = X(D(u_o) + \frac{X'}{1} D'_x(u_o) + \frac{X''}{2!} D''_{x^2}(u_o) + \dots + \frac{X^{(n-1)}}{(n-1)!} D^{n-1}_{x^{n-1}}(u_o) = 0 .$$

Since X is arbitrary, and this equation must be identically zero, it follows that

$$D(u_o) = D'_x(u_o) = D''_{x^2}(u_o) = \dots = D^{n-1}_{x^{n-1}}(u_o) \equiv 0,$$

which proves the assertion. Thus we see that the vanishing of the invariants plays the same role for (15) as it does for the second order equation.

LeRoux then considers the case when x is a multiple characteristic variable, so that (15) has the form

$$(26) \qquad D(z) = \Phi_o \frac{\partial^{n-m} z}{\partial x^{n-m}} + \Phi_1 \frac{\partial^{n-m-1} z}{\partial x^{n-m-1}} + \dots + \Phi_{n-m}\, z = 0 ,$$

where $\Phi_o, \Phi_1, \cdots$ designate the differential multipliers, e.g.

$$\Phi_o = \frac{\partial^r}{\partial y^r} + a_1 \frac{\partial^{r-1}}{\partial y^{r-1}} + a_2 \frac{\partial^{r-2}}{\partial y^r} + \dots \quad (r \leqq m).$$

The first substitution we have already considered, namely

$$z_1 = u_o \frac{\partial}{\partial y}\left(\frac{z}{u_o}\right).$$

We attempt to decompose $\Phi_o(z)$ into differential factors, so that $\Phi_o(z) = \alpha_1\alpha_2 \cdots \alpha_{r-1} u_o \frac{\partial}{\partial y}\frac{1}{\alpha_{r-1}}\frac{\partial}{\partial y}\frac{1}{\alpha_{r-2}} \cdots \frac{\partial}{\partial y}\frac{1}{\alpha_1}\frac{\partial}{\partial y}\frac{z}{u_o}$.

Then we define

$$z_1 = u_o \frac{\partial}{\partial y}\frac{z}{u_o},$$

$$z_2 = \alpha_1 u_o \frac{\partial}{\partial y}\frac{1}{\alpha_1}\frac{\partial}{\partial y}\frac{z}{u_o} = \alpha_1 u_o \frac{\partial}{\partial y}\frac{z}{\alpha_1 u_o},$$

$$z_3 = \alpha_1\alpha_2 u_o \frac{\partial}{\partial y}\frac{1}{\alpha_2}\frac{\partial}{\partial y}\frac{1}{\alpha_1}\frac{\partial}{\partial y}\frac{z}{u_o} = \alpha_1\alpha_2 u_o \frac{\partial}{\partial y}\frac{z_2}{\alpha_1\alpha_2 u_o},$$

$$\cdots$$

and using these we are able to reduce the equation. If $\Phi_o(z)$ does not have such a decomposition, we may still reduce the equations by a method which will be described briefly. The proof of the following is rather lengthy and detailed and does not bear repeating here. It may be found, however in LeRoux's work previously noted (footnote 22).

Let $u_o, u_1, u_2, \ldots$ denote a set of linearly independent integrals of the equations (18) which define the conditions on the coefficients for the existence of a solution of the form of (16). We then define the functions

$$k_1 = \frac{1}{u_o^2}\begin{vmatrix} u_o & \frac{\partial u_o}{\partial y} \\ u_1 & \frac{\partial u_1}{\partial y} \end{vmatrix}, \qquad k_2 = \frac{1}{u_o^3}\begin{vmatrix} u_o & \frac{\partial u_o}{\partial y} & \frac{\partial^2 u_o}{\partial y^2} \\ u_1 & \frac{\partial u_1}{\partial y} & \frac{\partial^2 u_1}{\partial y^2} \\ u_2 & \frac{\partial u_2}{\partial y} & \frac{\partial^2 u_2}{\partial y^2} \end{vmatrix}, \ldots$$

Set
$$l_i = \frac{k_i k_{i-2}}{k_{i-1}^2} .$$

The equation
$$\frac{\partial}{\partial y}\,\frac{1}{l_i}\,\frac{\partial}{\partial y}\,\frac{1}{l_{i-1}}\,\cdots\,\frac{\partial}{\partial y}\,\frac{1}{l_1}\,\frac{\partial}{\partial y}\left(\frac{u}{u_o}\right)=0$$
admits as solutions the functions $u_o, u_1, \ldots u_i$. Then the set of transformations is easily defined by setting

$$z_1 = u_o \frac{\partial}{\partial y}\left(\frac{z}{u_o}\right)$$

$$z_2 = u_o l_1 \frac{\partial}{\partial y}\,\frac{1}{l_1}\,\frac{\partial}{\partial y}\left(\frac{z}{u_o}\right) = u_o l_1 \frac{\partial}{\partial y}\left(\frac{z_1}{u_o l_1}\right)$$
$$\cdots$$
$$z_i = u_o l_1 l_2 \cdots l_{i-1}\,\frac{\partial}{\partial y}\,\frac{1}{l_{i-1}}\,\frac{\partial}{\partial y}\,\frac{1}{l_{i-2}}\cdots\frac{\partial}{\partial y}\,\frac{1}{l_1}\,\frac{\partial}{\partial y}\left(\frac{z}{u_o}\right).$$

If there exists a particular integral of the form of Euler, the invariants l will become zero after a certain number of iterations, and the chain of transformations will stop.

C. Following the work of Darboux and LeRoux, Dini[23] made an extension of the Laplace method to the linear second order equation in an arbitrary number of independent variables. Dini considered the equation

$$(27)\qquad \sum_{i,j=1}^{n} A_{ij}\,\frac{\partial^2 z}{\partial x_i \partial x_j} + \sum_{j=1}^{n} G_j\,\frac{\partial z}{\partial x_j} + Nz + H = 0, \quad A_{ij} \equiv A_{ji},$$

and posed the question, can we transform (27), into the form

$$(28)\qquad \sum_{i=1}^{n} k_i\,\frac{\partial \Theta}{\partial x_i} + M\Theta + \sum_{i=1}^{n} \alpha_i\,\frac{\partial z}{\partial x_i} + Lz + H = 0,$$

where

$$(29) \qquad \Theta = \sum_{j=1}^{n} a_j \frac{\partial z}{\partial x_j} + bz ,$$

and the functions $a_1, a_2, \ldots, a_n, b, k_1, \ldots k_n, M$ are $(2n+2)$ functions to be chosen arbitrarily? If we assume that this has been accomplished, substitute (29) into (28), and equate coefficients with the left side of (27), we obtain the $\frac{(n+1)(n+2)}{2}$ equations

$$(30) \qquad k_i a_j + k_j a_i = 2A_{ij} , \qquad i,j = 1, \ldots, n ,$$

$$(31) \qquad k_i b + a_i M + \sum_{j=1}^{n} k_j \frac{\partial a_i}{\partial x_j} + \alpha_i = G_i , \qquad i = 1, \ldots, n ,$$

$$(32) \qquad Mb + \sum_{j=1}^{n} k_j \frac{\partial b}{\partial x_j} + L = N ,$$

for the $(2n+2)$ unknowns. We observe, however, that in (30), the k_i and a_j appear only as products $k_i a_j$, and not all of them are zero if (27) is to be effectively of second order. Thus we may select arbitrarily one k, say k_r, and set $k_r = 1$, without affecting these relations; and hence there are actually only $2n-1$ unknowns a_i and k_i. If we consider first the equations (30), we note that for $n = 2$, these are 3 equations in 3 unknowns, and this system is determined. For $n \geqq 3$, however, the system is overdetermined, and one would expect to have $\frac{(n-1)(n-2)}{2}$ relations among the coefficients A_{ij} so that only $2n-1$ of the equations (30) will be independent.

We see then, that before we may even begin to seek apply a "Laplace method" to equation (27), the coefficients A_{ij} must satisfy $\frac{(n-1)(n-2)}{2}$ conditions. Dini showed that in general these conditions may be stated as

$$(33) \quad A_{hr}A_{hs} - A_{hh}A_{rs} = \varepsilon_r \varepsilon_s \sqrt{A_{hr}^2 - A_{hh}A_{rr}} \sqrt{A_{hs}^2 - A_{hh}A_{ss}}\,, \quad r,s = 2,3,\ldots,n, \; r \neq s,$$

where $\varepsilon_r = \pm 1$, $\varepsilon_s = \pm 1$, and these signs are determined by the sign of the radicals of the roots of the quadratic equation

$$A_{11}\lambda_r^2 - 2A_{1r}\lambda_r + A_{rr} = 0 .$$

Here A_{hh} is considered to be the first non-vanishing coefficient A_{ii}, $i=1,\ldots,n$. If all the A_{ii} are zero, the equations (33) are satisfied identically for all A_{ii}, $i=1,\ldots,n$.

The classification of the various allowable types of equation (27) becomes the next important consideration. We list here without proof some of the more important properties of (27) derived by Dini. First, if $n > 2$, and all coefficients of (27) are real valued, then for any pair of variables (x_r,x_s) for which the partial differential equation is not parabolic, the equation must always be of the same type. That is, ignoring all pairs of variables for which the equation is parabolic, and considering all remaining pairs of variables, the equation must always be elliptic, or always be hyperbolic type, with respect to each pair of variables. Another property, true even for complex-valued coefficients: if $A_{hh} \neq 0$, and the partial differential equation (27) is parabolic with respect to the pairs (x_h,x_r) and (x_h,x_s), then it will also be parabolic with respect to the pair (x_r,x_s). These properties follow from the conditions (33).

Dini then defines the following terminology. Equation (27) is said to be of parabolic type provided that it is parbolic with respect to every pair of variables (x_r,x_s); if there exists at least one pair of

variables (x_r, x_s) such that (27) is not parabolic with respect to (x_r, x_s) then (27) is said to be of elliptic (hyperbolic) type if it is elliptic (hyperbolic) with respect to every pair of variables (x_r, x_s) for which it is not of parabolic type; otherwise (27) is said to be of mixed type. Then Dini proves that if (27) is not of parabolic type, then either there exist exactly two systems of values of k and a which satisfy (30), or there exists none, while if (27) is of parabolic type, there exist at most one system of values of k and a satisfying (30). In the first case, one obtains one system of values from the other merely by interchanging the k's and the a's. Denoting these values by (k,a) and (a,k) respectively, we may call them conjugate sets.

Now our first thought in attempting to reduce (27) to the integration of two partial differential equations of first order, is that perhaps (28) will reduce to a first order equation in Θ alone. In addition to the $\frac{(n-1)(n-2)}{2}$ conditions previously noted as necessary to reduce (27) to (28) and (29), this requires n-1 further conditions to insure that $\alpha_1 = \alpha_2 = \ldots = \alpha_n = L = 0$. These conditions arise from the $(n+1)$ equations (31) and (32) which determine only the two unknowns b and M. Assuming that the equation is not of parabolic type and hence, without loss of generality, that it is not parabolic with respect to the pair $(x_1 x_2)$, Dini shows that the n-1 conditions can be formulated as

$$N - Mb - B = 0 \tag{34}$$

and

$$(35) \qquad \begin{vmatrix} A_{11} & A_{21} & G_1 - A_1 \\ A_{12} & A_{22} & G_2 - A_2 \\ A_{1s} & A_{2s} & G_s - A_s \end{vmatrix} = 0 \quad , \quad s = 3, 4, \ldots, n,$$

where $A_s = \sum_{r=1}^{n} k_r \frac{\partial a_s}{\partial x_r}$, and $B = \sum_{r=1}^{n} k_r \frac{\partial b}{\partial x_r}$.

We may look upon L and the α_i in (28) as playing the role of the invariants. Thus to say that the conditions (34) and (35) are satisfied, is to say that the invariants of (27) are all zero. If we denote $D^*(u) = M_u + \sum_{j=1}^{n} k_j \frac{\partial u}{\partial x_j}$, these invariants may be written as

$$(36) \qquad \begin{aligned} L &= N - D^*(b), \\ \alpha_i &= G_i - k_i b - D^*(a_i). \end{aligned}$$

In order to start the chain of equations when these invariants are not all zero, Dini imposes conditions similar to those imposed by us on the third order equations of Section IV. For equation (27) to be cascaded, Dini requires that conditions (35) be satisfied. That is to say, the invariants α_i are all zero, but the invariant L is not. Then equation (28) may be solved for z, in terms of Θ and its first partial derivatives. This expression is substituted into (29) and a new second order equation, for Θ, is obtained. Not only will this equation be of the same form as (27), but the second order coefficients A_{ij} are reproduced. Hence our new equation is

$$(37) \qquad \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 \Theta}{\partial x_i \partial x_j} + \sum_{j=1}^{n} G_j' \frac{\partial \Theta}{\partial x_j} + N'\Theta + H' = 0 .$$

The new coefficients are given by

$$G_s' = G_s + K_s - A_s - k_s L_a ,$$

$$(38) \qquad N' = N + \sum a_r \frac{\partial M}{\partial x_r} - ML_a - B ,$$

$$H' = H(b + H_a - L_a) ,$$

where $K_s = \sum a_r \frac{\partial k_s}{\partial x_r}$, and H_a, L_a are given by

$$\Phi_a = \sum_{r=1}^{n} a_r \frac{\partial \log \Phi}{\partial x_r} .$$

Since the relations (30) used to determine (k,a) for (27) involve only the coefficients A_{ij}, it immediately follows that if there exists a set (k,a) which splits (27) into (28) and (29), the same set (k,a) will suffice to split (37) analogously. It will be necessary however to compute a new M' and b', and a new set of invariants L' , and α_i' . The conditions under which the α_i' are zero become

$$(39) \qquad \begin{vmatrix} A_{11} & A_{21} & G_1' - A_1 \\ A_{12} & A_{22} & G_2' - A_2 \\ A_{13} & A_{2s} & G_s' - A_s \end{vmatrix} = 0 \quad , \text{ for } s = 3,4,\ldots,n.$$

which are quite similar to the conditions (35). If these conditions are satisfied, and further $L' = N' - M'b' - B' = 0$, then (37) reduces to two first order equations as desired. If L' is not zero, however we must again require that conditions (39) be satisfied before we can iterate the process to obtain a third second order equation of the form (27).

Dini's very remarkable result, however, is the following. If the conditions (35) and the conditions (39) are all satisfied, then we need impose no further condtions to iterate the process indefinitely. That is to say, if the invariants α_i and α_i' are all zero, then all invariants $\alpha_i^{(m)}$ will be zero for any positive number m of iterations. This result becomes evident if we compute, say, G_s'' by the rule given in (38) for G_s', substitute $G_1'' - A_1$, $G_2'' - A_2$, and $G_s'' - A_s$ for the corresponding terms in (39), and employ the rule for evaluating a determinant when one of the columns is a sum of two or more columns. Each of the resulting determinants in the sum must vanish as a direct consequence of conditions (35) and (39).

In summary then, in order for the equation (27) to be cascaded as many times as necessary for the L invariant to vanish, there must be satisfied a total of

$$\frac{(n-1)(n-2)}{2} + 2\,(n-2) = \frac{(n+3)(n-2)}{2}$$

conditions on the coefficients. The first $\frac{(n-1)(n-2)}{2}$ conditions are necessary to reduce (27) to two first order equations, while the remaining $2(n-2)$ are necessary to permit the cascading to continue. We observe that for $n = 2$, no conditions are necessary, and this is the original case

considered by Laplace, Legendre and Darboux.

D. Finally, mention should be made of the work of Burgatti[24] who considered the equation of elliptic type,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0. \tag{40}$$

Burgatti found that the two expressions

$$\begin{aligned} H &= \frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}, \\ K &= \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{a^2 + b^2}{2} - 2c \end{aligned} \tag{41}$$

are invariant relative to the transformation $z = \lambda z'$. Further, if H and K are both zero, equation (40) may be reduced to

$$\frac{\partial^2 z'}{\partial x^2} + \frac{\partial^2 z'}{\partial y^2} = 0,$$

which is Laplace's equation; if H is zero but K is not zero, then (40) takes the form

$$\frac{\partial^2 z'}{\partial x^2} + \frac{\partial^2 z'}{\partial y^2} + c'z' = 0;$$

while if K is zero, but H is not zero, (40) becomes

$$\beta\frac{\partial^2(\alpha z')}{\partial x^2} + \alpha\frac{\partial^2(\beta z')}{\partial y^2} = 0,$$

where α and β are functions of x and y.

In the notation of Section V, let

$$D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c,$$

then

$$D_x' = 2\frac{\partial}{\partial x} + a$$

$$D_y' = 2\frac{\partial}{\partial y} + b$$

and the invariants (41) then become

$$(42) \qquad H = \frac{D_y'(a) - D_x'(b)}{2} = \left(\frac{D_y'(a)}{2} - c\right) - \left(\frac{D_x'(b)}{2} - c\right)$$

$$K = \frac{D_x'(a) + D_y'(b)}{2} - 2C = \left(\frac{D_x'(a)}{2} - c\right) + \left(\frac{D_y'(b)}{2} - c\right)$$

Since the change of variable and the vanishing of invariants does not reduce the equation to two first order equations, no consideration can be given to cascading the equations of elliptic type utilizing the invariants H and K. The work of Dini shows us, however, that this equation can be cascaded if we admit the possibility of complex coefficients.

FOOTNOTES

1. Laplace, Perre Simon; Marquis de; "Recherche sur la calcul integral aux différences partielles," Oeurres IX, pp 5-68.

   See also: Memoires de l'Academie Royale des Science de Paris, 1773 (Published 1777).

2. Darboux, Gaston; Lecons sur la Théorie Génerale Des Surfaces, Vol 2, Chapter II, pp 23-53.

3. Dini, Ulisse; "Sopre Una Classe di Equazioni A Derivate Parziali di Second Ordine Con Un Numèro Qualunque di Variable," Opere, Vol III, pp 489-566, Rome, 1955.

   See also: Atti Acc. Naz. dei Lincei, Mem. Classe Sc., fis., mat., nat (5), 4(1901); pp 121-178.

4. LeRoux, J.; "Extension de la Méthode de Laplace aux Equations lineaires aux Derivées Partielles d'Ordre Superieur au Second", Bulletin de la Société Mathématique de France, 27 (1899); pp 237-262.

5. This is not done in the usual manner (c.f. G. Darboux, op. cit.).

The exponential integrating factor is used here, rather than solving the second equation of (6a') directly for u, because in the third order extension (Section IV), and the n^{th} order extension (Section V) we will encounter the equation with u and numerous partial derivatives. These equations cannot be solved directly for u, but the integrating factor method can be employed.

6. Volterra, Vito ; "Sulle equazione differenziali lineari", Rend. dei Lincei 3(1887) pp 391-396.

   See also: "Sui fondamente della teoria della equazioni differenziali lineari", Mem. Soc. Ital. Sc III 8(1887), p 6.

7. See, however, Rasch, G., "Zur Theorie und Anwendung des Productintegrals", Journal für die reine und angewandte Mathematik, Vol 171 (1934), pp 65-119. On page 78 of this article, Rasch discusses this problem. His derivation is quite similar to the author's, but he does not carry through to the final form of equation (9) of this section.

8. Schlesinger, Ludwig; "Beitrage zur Theorie der Systeme linearer homogener Differentialgleichungen," Jour. reine und angewandte Math., Vol 128 (1905) pp 263-297.

9. ibid.

10. ibid.

11. Schlesinger, Ludwig; "Neue Grundlagen für einen Infinitesimalkalkul der Matrizen," Math. Zeitschrift, Vol 33 (1931), pp 33-61. See also: "Weitere Beiträge zum Infinitesimalkalkul des Matrizen," Math Zeitschrift, Vol 35 (1932) pp 485-501.

12. In this section we have used column matrices in lieu of the row matrices of Section II, merely for the sake of appearance. It was desired to keep the results in as close analogy as possible with those of Darboux obtained with the single second order equation. The mechanics are quite the same whether using row or column matrices. If desired, of course, square matrices with appropriate zeroes can be used in lieu of either row or column matrices.

13. Wedderburn, Joseph H. M.; Lectures on Matrices, New York, AMS, 1934, p 128.

14. Wedderburn, J. H. M.; op.cit. p 116.

15. cf. Webster, Arthur G; Partial Differential Equations of Mathematical Physics, New York, G. E. Stechert & Co., 1933, pp 253-255.

16. Wedderburn, J. H. M; op cit. p 122.

17. Darboux, G.; op cit pp 31-32.

18. Darboux, G; op cit.

19. Le Roux, J.; "Sur les Integrales Des Équation Linéaires aux Dérivées Partielles du Second Ordre A Deux Variables Indépendantes", *Annales Scientifiques de L'Ecole Normale Supérieure*, III, 12 (1895), p 242.

20. Darboux, G.; op.cit. p.28.

21. Darboux, G.; *Lecons sur la Théorie Générale Des Surfaces*, Vol 4, pp 267-286.

22. Le Roux, J.; "Extension de la Méthode de Laplace Aux Équations Linéaires Aux Dérivées Partielles d'Ordre Supérieur Au Second," *Bulletin de la Société Mathématique de France*, Vol 27 (1899); pp 237-262.

23. Dini, Ulisses; *Atti Acc Naz dei Lincei Mem Classe Sc., fis mat., mat.*, (5) 4 (1901); pp 121-178.
See also "Sopra Una Classe di Equazioni A Derivate Parziali di Second Ordine Con Un Numero Qualunque di Variabli," *Opere*, Vol III (1955), Rome; pp 489-566.

24. Burgatti, P.; "Sull' equazione lineari alle derivate parziali del 2nd ordine (tipo ellittico), e sopra una classificazione dei sisteme di linee ortogonali che si possono tracciare sopra una superficie", Anali di Matematica (2) XXIII; pp 225-267.

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